Weak Bases of Boolean Co-Clones

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Abstract. Universal algebra has proven to be a useful tool in the study of constraint satisfaction problems (CSP) since the complexity, up to logspace reductions, is determined by the clone of the constraint language. But two CSPs corresponding to the same clone may still differ substantially with respect to worst-case time complexity, which makes clones ill-suited when comparing running times of CSP problems. In this article we instead consider an algebra where each clone splits into an interval of strong partial clones such that a strong partial clone corresponds to the CSPs that are solvable within the same $O(c^n)$ bound. We investigate these intervals and give relational descriptions, weak bases, of the largest elements. They have a highly regular form and are in many cases easily relatable to the smallest members in the intervals, which suggests that the lattice of strong partial clones has a simpler structure than the lattice of partial clones.

1 Introduction

A set of functions is called a *clone* if (1) it is closed under composition of functions and (2) it contains all projection functions of the form $e_i^n(x_1, \ldots, x_n) = x_i$. Dually, a set of relations is called a *relational clone*, or a *co-clone*, if it contains all relations definable through formulas built up from existential quantification, conjunction, and equality constraints, over the set in question. Clones and co-clones thus group together functions and relations which share some fundamental properties, and to better understand the structure of the full set one often considers restricted sets, *bases*, which are still expressive enough to preserve all properties of the full set. For any domain it is thus of interest to classify the clones and co-clones on that domain and obtain a better understanding of its lattice ordered by set inclusion. In the Boolean case this goal was achieved by Post [11] and the lattice of Boolean clones is hence known as *Post's lattice*. Essentially the lattice determines the expressive properties of all possible Boolean functions. Due to the *Galois connection* between clones and co-clones the lattice of Boolean co-clones is anti-isomorphic to Post's lattice and therefore works as a complete classification of all Boolean languages.

This means that given a set of relations one can associate a clone which mirrors its structure. Note however that the ordering between the two lattices is reversed and hence the smallest co-clone in fact corresponds to the largest clone. Intuitively this holds because a small co-clone has a large associated clone. The reader is referred to Böhler et al. [3,4] for a list of bases of Boolean clones and co-clones. The lattice of Boolean co-clones is visualized in Figure 1. The complexity of various computational problems parameterized by constraint languages such as the *constraint satisfaction problem* (CSP) has been shown to be determined up to logspace reducibility by Post's lattice [2,7]. By constraint language we here understand any finite set of Boolean relations. If one on the other hand is interested in complexity classifications based on reductions which preserve the structure of instances to a larger degree, e.g. the number of variables, Post's lattice falls short, since even logspace reductions may introduce new variables which affect the running-time.

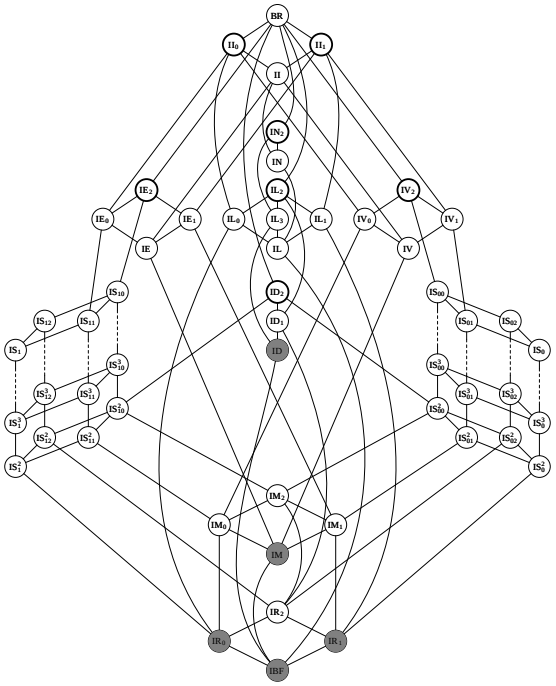

Fig. 1. The lattice of Boolean co-clones. The co-clones which are covered by a single weak partial co-clone are colored in grey.

To remedy this a more fine-grained framework which further separates constraint languages based on their expressive properties is necessary. In Jonsson et al. [8] the lattice of *strong partial clones* is demonstrated to have the required properties. By this we mean that constraint languages corresponding to the same strong partial clone result in CSP problems solvable within exactly the same $O(c^n)$ bound. Hence a classification of the lattice of strong partial clones similar to that of Post's lattice would provide a powerful and nuanced framework for studying complexity of CSP and related problems. We wish to emphasize that even though the lattice of partial clones is known to be uncountable [1] the same does not necessarily hold for the lattice of strong partial clones. Ideally, for each clone C, one would like to determine the interval of strong partial clones whose subset of total functions equals C. The strong partial clones in this interval are said to *cover* C. Even though a complete classification appears difficult a good starting point is to consider the endpoints of each interval, i.e. the largest and smallest strong partial clone corresponding to C. In Creignou et al. [5] relational descriptions known as *plain bases* of the smallest members of these intervals are given. In this article we give simple relational descriptions known as *weak bases* of the largest elements in these intervals. Our work builds on the result of Schnoor and Schnoor [13,14] but differs in two important aspects: first, each weak base presented can in a natural sense be considered to be minimal; second, we present alternative proofs where Schnoor's and Schnoor's procedure results in relations which are exponentially larger than the bases given by Böhler et al. [4] and Creignou et al. [5], and are thus also able to cover the infinite chains in Post's lattice.

Due to the Galois connection between clones and co-clones the weak bases constitute the least expressive languages, and as such each weak base results in a CSP problem with the property that it is solvable at least as fast as any other CSP problem within the same co-clone [8]. Hence the weak bases presented in Section 3 are closely connected to upper bounds of running times for problems parameterized by constraint languages.

2 Preliminaries

In this section we introduce some basic notions from universal algebra necessary for the construction of weak bases. If f is an n-ary function and R an m-ary relation it is possible to extend f such that $f(t_1, \ldots, t_n) = \big(f(t_1[1], \ldots, t_n[1]), \ldots, f(t_1[m], \ldots, t_n[m])\big)$, where $t_i[j]$ denotes the j-th element of $t_i \in R$. If R is closed under f we say that f *preserves* R or that

f is a *polymorphism* of R. For a set of functions F we define $\mathrm{Inv}(\mathsf{F})$ (often abbreviated as IF) to be the set of all relations preserved by all functions in F. Dually we define $\mathrm{Pol}(\Gamma)$ for a set of relations Γ to be the set of polymorphisms of Γ. Sets of the form $\mathrm{Pol}(\Gamma)$ are referred to as *clones* and $\mathrm{Inv}(\mathsf{F})$ as *co-clones*. As a shorthand we let $\langle\Gamma\rangle = \mathrm{Inv}(\mathrm{Pol}(\Gamma))$ and $[\mathsf{F}] = \mathrm{Pol}(\mathrm{Inv}(\mathsf{F}))$. The sets Γ and F are called *bases* of $\langle\Gamma\rangle$ and $[\mathsf{F}]$, respectively. The set $[\mathsf{F}]$ is then the smallest set containing all projection functions and all functions obtained by composing functions from F, while $\langle\Gamma\rangle$ is equivalent to the smallest set containing all relations R *p.p. definable* over Γ, i.e. definitions of the form $R(x_1,\ldots,x_n) \equiv \exists y_1,\ldots,y_m \,.\, R_1(\mathbf{x_1}) \wedge \ldots \wedge R_k(\mathbf{x_k})$, where each $R_i \in \Gamma \cup \{\mathrm{Eq}\}$, each $\mathbf{x_i}$ is a vector over $x_1,\ldots,x_n$, $y_1,\ldots,y_m$, and $\mathrm{Eq} = \{(0,0),(1,1)\}$. Thus $[\cdot]$ and $\langle\cdot\rangle$ indeed capture the informal definitions of clones and co-clones given in the introduction. Moreover we have the *Galois connection* between clones and co-clones normally presented as:

Theorem 1 ([6]). *Let Γ and Δ be two sets of relations. Then $\langle\Gamma\rangle \subseteq \langle\Delta\rangle$ if and only if $\mathrm{Pol}(\Delta) \subseteq \mathrm{Pol}(\Gamma)$.*

To extend these notions to the case of partial clones we need some additional notation. If R is an n-ary Boolean relation and Γ a constraint language we say that R has a *quantifier-free primitive positive* (q.p.p.) implementation in Γ if $R(x_1,\ldots,x_n) \equiv R_1(\mathbf{x_1}) \wedge \ldots \wedge R_k(\mathbf{x_k})$, where each $R_i \in \Gamma \cup \{\mathrm{Eq}\}$ and each $\mathbf{x_i}$ is a vector over $x_1,\ldots,x_n$. We use $\langle\Gamma\rangle_{\not\exists}$ to denote the smallest set of relations closed under q.p.p. definability. If $\Gamma = \langle\Gamma\rangle_{\not\exists}$ then we say that Γ is a *weak partial co-clone*. We use the term weak partial co-clone to avoid confusion with partial co-clones used in other contexts (see Chapter 20.3 in Lau [10]). To get a corresponding concept on the functional side we extend the previous definition of a polymorphism and say that a partial function f is a *partial polymorphism* of a relation R if R is closed under f for every sequence of tuples for which f is defined. A set of (partial) functions C is said to be a *partial clone* if it contains all projection functions and is closed under composition of functions. A partial clone C is *strong* if for every n-ary $f \in \mathsf{C}$, C also contains all n-ary *subfunctions* g of f such that if $f(x_1,\ldots,x_n)$ for $(x_1,\ldots,x_n) \in \{0,1\}^n$ is defined then either $g(x_1,\ldots,x_n) = f(x_1,\ldots,x_n)$, or $g(x_1,\ldots,x_n)$ is undefined. By $\mathrm{pPol}(\Gamma)$ we denote the set of partial polymorphisms of the set of relations Γ. Sets of the form $\mathrm{pPol}(\Gamma)$ always form strong partial clones and again we have a Galois connection between clones and co-clones.

Theorem 2 ([12]). *Let Γ and Δ be two sets of relations. Then $\langle\Gamma\rangle_{\not\exists} \subseteq \langle\Delta\rangle_{\not\exists}$ if and only if $\mathrm{pPol}(\Delta) \subseteq \mathrm{pPol}(\Gamma)$.*

For a co-clone IC we define $\mathcal{I}(\mathsf{IC}) = \{\mathsf{IC}' \mid \mathsf{IC}' = \langle \mathsf{IC}' \rangle_{\not\exists}$ and $\langle \mathsf{IC}' \rangle = \mathsf{IC}\}$. In other words $\mathcal{I}(\mathsf{IC})$ is the interval of all weak partial co-clones occurring inside of IC. We refer to $\mathcal{I}(\mathsf{IC})$ as an interval because whenever a weak partial co-clone lies between two other weak partial co-clones included in $\mathcal{I}(\mathsf{IC})$, then this weak partial co-clone is also included in $\mathcal{I}(\mathsf{IC})$. Let $\mathcal{I}_{\cap}(\mathsf{IC}) = \bigcap_{\mathsf{IC}' \in \mathcal{I}(\mathsf{IC})} \mathsf{IC}'$. To be consistent with Schnoor's and Schnoor's [13] notation which is defined in terms of clones instead of co-clones we also define $\mathcal{I}_{\cup}(\mathsf{C}) = \bigcup_{\mathsf{IC}' \in \mathcal{I}(\mathsf{IC})} \mathrm{pPol}(\mathsf{IC}')$ for a clone C. The set $\mathcal{I}_{\cup}(\mathsf{C})$ is the union of all strong partial clones covering C, from which it follows that $\mathrm{pPol}(\mathcal{I}_{\cap}(\mathsf{IC})) = \mathcal{I}_{\cup}(\mathsf{C})$.

Definition 3. *Let* C *be a clone. A constraint language* Γ *is a* weak base *of* IC *if* $\mathrm{pPol}(\Gamma) = \mathcal{I}_{\cup}(\mathsf{C})$.

Implicitly Γ is also a base of IC since $\langle \Gamma \rangle_{\not\exists} \in \mathcal{I}(\mathsf{IC})$, and due to the Galois connection, it is also the smallest element in $\mathcal{I}(\mathsf{IC})$ [13]. The following theorem is immediate from the definition and the fact that $\mathrm{pPol}(\mathcal{I}_{\cap}(\mathsf{IC})) = \mathcal{I}_{\cup}(\mathsf{C})$.

Theorem 4 ([13]). *Let* C *be a clone and* Γ *be a weak base of* IC*. Then* $\Gamma \subseteq \langle \Gamma' \rangle_{\not\exists}$ *for any base* Γ' *of* IC*.*

If R is an n-ary relation with $m = |R|$ elements we let the matrix representation of R be the $m \times n$-matrix containing the tuples of R as rows stored in lexicographical order. Note that the ordering is only relevant to ensure that the representation is unambiguous. Given a natural number n the 2^n-ary relation χ^n is the relation which contains all natural numbers from 0 to $2^n - 1$ as columns in the matrix representation. Thus χ^n contains the tuples $t_1, \ldots, t_n$ and for $1 \leq i \leq 2^n$ the tuple $(t_1[i], t_2[i], \ldots, t_n[i])$ is a binary representation of the natural number $i - 1$ where $t_1[i]$ is the most significant bit. For any clone C and relation R we define $\mathsf{C}(R)$ to be the relation $\bigcap_{R' \in \mathsf{IC}, R \subseteq R'} R'$, i.e. the smallest extension of R preserved by C. For a clone C we say that IC has *core-size* s if there exist relations R, R' such that $\mathrm{Pol(R)} = \mathsf{C}$, $R = \mathsf{C}(R')$ and $s = |R'|$. The relation R' is in this case said to be a C-*core* of R [13]. Minimal core-sizes for all Boolean co-clones have been identified by Schnoor [14]. We are now ready to state Schnoor's and Schnoor's [13] main result which effectively gives a weak base for any co-clone with a finite core-size.

Theorem 5 ([13]). *Let* C *be a clone and* s *be a core-size of* IC*. Then the relation* $\mathsf{C}(\chi^s)$ *is a weak base of* IC*.*

The disadvantage of the theorem is that relations of the form $\mathsf{C}(\chi^s)$ have exponential arity with respect to the core-size. We therefore introduce another measurement of minimality which ensures that a given relation is indeed minimal in the sense that it does not contain any superfluous columns and that there is no subset of the relation which is still a weak base. An n-ary relation R is said to be *irredundant* if there are no duplicate columns in the matrix representation and it is said to have a *fictitious argument* if there exists an $1 \leq i \leq n$ such that $(a_{j,1}, ..., a_{j,i}, \ldots, a_{j,n}) \in R$ if and only if $(a_{j,1}, ..., \overline{a_{j,i}}, \ldots, a_{j,n}) \in R$.

Definition 6. *A relation R is* minimal *if it is (1) irredundant, (2) contains no fictitious arguments and (3) if there is no $R' \subset R$ such that $\langle R \rangle = \langle R' \rangle$.*

Minimal weak bases have the property that they can be implemented without the use of the equality operator. If we let $\langle \cdot \rangle_{\exists \neq}$ denote the closure of q.p.p. definitions without equality we therefore get the following theorem.

Theorem 7 ([13]). *Let C be a clone and Γ be a minimal weak base of IC. Then, for any base Γ' of IC, it holds that $\Gamma \subseteq \langle \Gamma' \rangle_{\exists \neq}$.*

Hence minimal weak bases give the largest possible expressibility results and are applicable for problems such as counting CSP$(\cdot)$, where equality constraints in an instance may increase the number of solutions exponentially [13].

3 Minimal weak bases of all Boolean co-clones

In this section we proceed by giving minimal weak bases for all Boolean co-clones with finite core-size. The results are presented in Table 1. Each entry in the table consists of a co-clone, its minimal core-size, a minimal weak base and a base of the corresponding clone. As convention we use Boolean connectives to represent relations and functions whenever this promotes readability. For example $x_1 x_2$ denotes the relation $\{(1,1)\}$ while $x_1 \neq x_2$ denotes the relation $\{(0,1),(1,0)\}$. We use F for the relation $\{(0)\}$ and T for the relation $\{(1)\}$. The relations OR^n and NAND^n are n-ary *or* and *nand*. EVEN^n is the n-ary relation which holds if the sum of its arguments is even, and conversely for ODD^n. By $R^{1/3}$ we denote the 3-ary relation $\{(0,0,1),(0,1,0),(1,0,0)\}$. If R is an n-ary relation we often use $R_{m\neq}$ to denote the $(n+m)$-ary relation defined as

$R_{m\neq}(x_1,\ldots,x_{n+m}) \equiv R(x_1,\ldots,x_n) \wedge (x_1 \neq x_{n+1}) \wedge \ldots \wedge (x_n \neq x_{n+m})$. Variables are named $x_1,\ldots,x_n$ or x except when they occur as arguments to F or T in which case they are named c_0 and c_1 respectively to indicate that they are constants.

For the co-clones $\mathsf{IR_2}$, IM, ID, $\mathsf{ID_1}$, IL, $\mathsf{IL_0}$, $\mathsf{IL_1}$, $\mathsf{IL_2}$, $\mathsf{IL_3}$, IV, $\mathsf{IV_0}$, IE, $\mathsf{IE_1}$, IN, $\mathsf{IN_2}$, II, $\mathsf{II_0}$, $\mathsf{II_1}$ and BR, the result follows immediately from Theorem 5, the minimal core-sizes for each co-clone, and a suitable rearrangement of arguments. The problem of checking whether an n-ary relation R generates a co-clone can be checked in time $O(n^2|R|)$ using the algorithm in Creignou et al. [5], and through exhaustive search, i.e. by repeatedly removing redundant columns and tuples, one can verify that the bases are also minimal. This has been done by a computer program which is available at an online repository [9].

Table 1. Weak bases for all Boolean co-clones with a finite base. The rightmost column contains a base of the corresponding clone, where $\mathrm{id}(x) = x$ and $h_n(x_1,\ldots,x_{n+1}) = \bigvee_{i=1}^{n+1} x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1}$, $\mathrm{dual}(f)(a_1,\ldots,a_n) = 1 - f(\overline{a_1},\ldots,\overline{a_n})$.

Co-clone	Core-size	Weak base	Base of clone
IBF	1	$\mathrm{Eq}(x_1, x_2)$	$\{x_1 \wedge x_2, \overline{x_1}\}$
$\mathsf{IR_0}$	1	$\mathrm{F}(c_0)$	$\{x_1 \wedge x_2, x_1 \oplus x_2\}$
$\mathsf{IR_1}$	1	$\mathrm{T}(c_1)$	$\{x_1 \vee x_2, x_1 \oplus x_2 \oplus 1\}$
$\mathsf{IR_2}$	1	$\mathrm{F}(c_0) \wedge \mathrm{T}(c_1)$	$\{x_1 \vee x_2, x_1 \wedge (x_2 \oplus x_3 \oplus 1)\}$
IM	1	$(x_1 \to x_2)$	$\{x_1 \vee x_2, x_1 \wedge x_2, 0, 1\}$
$\mathsf{IM_0}$	2	$(x_1 \to x_2) \wedge \mathrm{F}(c_0)$	$\{x_1 \vee x_2, x_1 \wedge x_2, 0\}$
$\mathsf{IM_1}$	2	$(x_1 \to x_2) \wedge \mathrm{T}(c_1)$	$\{x_1 \vee x_2, x_1 \wedge x_2, 1\}$
$\mathsf{IM_2}$	3	$(x_1 \to x_2) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)$	$\{x_1 \vee x_2, x_1 \wedge x_2\}$
$\mathsf{IS}_0^n, n \geq 2$	n	$\mathrm{OR}^n(x_1,\ldots,x_n) \wedge \mathrm{T}(c_1)$	$\{x_1 \to x_2, \mathrm{dual}(h_n)\}$
$\mathsf{IS}_{02}^n, n \geq 2$	n	$\mathrm{OR}^n(x_1,\ldots,x_n) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)$	$\{x_1 \vee (x_2 \wedge \overline{x_3}), \mathrm{dual}(h_n)\}$
$\mathsf{IS}_{01}^n, n \geq 2$	n	$\mathrm{OR}^n(x_1,\ldots,x_n) \wedge (x \to x_1 \cdots x_n) \wedge \mathrm{T}(c_1)$	$\{\mathrm{dual}(h_n), 1\}$
$\mathsf{IS}_{00}^n, n \geq 2$	$\max(3, n)$	$\mathrm{OR}^n(x_1,\ldots,x_n) \wedge (x \to x_1 \cdots x_n) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)$	$\{x_1 \vee (x_2 \wedge x_3), \mathrm{dual}(h_n)\}$
$\mathsf{IS}_1^n, n \geq 2$	n	$\mathrm{NAND}^n(x_1,\ldots,x_n) \wedge \mathrm{F}(c_0)$	$\{x_1 \wedge \overline{x_2}, h_n\}$
$\mathsf{IS}_{12}^n, n \geq 2$	n	$\mathrm{NAND}^n(x_1,\ldots,x_n) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)$	$\{x_1 \wedge (x_2 \vee \overline{x_3}), h_n\}$
$\mathsf{IS}_{11}^n, n \geq 2$	n	$\mathrm{NAND}^n(x_1,\ldots,x_n) \wedge (x \to x_1 \cdots x_n) \wedge \mathrm{F}(c_0)$	$\{h_n, 0\}$
$\mathsf{IS}_{10}^n, n \geq 2$	$\max(3, n)$	$\mathrm{NAND}^n(x_1,\ldots,x_n) \wedge (x \to x_1 \cdots x_n) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)$	$\{x_1 \wedge (x_2 \vee x_3), h_n\}$
ID	1	$(x_1 \neq x_2)$	$\{(x_1 \wedge \overline{x_2}) \vee (x_1 \wedge \overline{x_3}) \vee (\overline{x_2} \wedge \overline{x_3})\}$
$\mathsf{ID_1}$	2	$(x_1 \neq x_2) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)$	$\{(x_1 \wedge x_2) \vee (x_1 \wedge \overline{x_3}) \vee (x_2 \wedge \overline{x_3})\}$
$\mathsf{ID_2}$	3	$\mathrm{OR}^2_{2\neq}(x_1, x_2, x_3, x_4) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)$	$\{h_2\}$
IL	2	$\mathrm{EVEN}^4(x_1, x_2, x_3, x_4)$	$\{x_1 \oplus x_2, 1\}$
$\mathsf{IL_0}$	2	$\mathrm{EVEN}^3(x_1, x_2, x_3) \wedge \mathrm{F}(c_0)$	$\{x_1 \oplus x_2\}$
$\mathsf{IL_1}$	2	$\mathrm{ODD}^3(x_1, x_2, x_3) \wedge \mathrm{T}(c_1)$	$\{x_1 \oplus x_2 \oplus 1\}$
$\mathsf{IL_2}$	3	$\mathrm{EVEN}^3_{3\neq}(x_1,\ldots,x_6) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)$	$\{x_1 \oplus x_2 \oplus x_3\}$
$\mathsf{IL_3}$	3	$\mathrm{EVEN}^4_{4\neq}(x_1,\ldots,x_8)$	$\{x_1 \oplus x_2 \oplus x_3 \oplus 1\}$
IV	2	$(\overline{x_1} \leftrightarrow \overline{x_2}\,\overline{x_3}) \wedge (\overline{x_2} \vee \overline{x_3} \to \overline{x_4})$	$\{x_1 \vee x_2, 0, 1\}$
$\mathsf{IV_0}$	2	$(\overline{x_1} \leftrightarrow \overline{x_2}\,\overline{x_3}) \wedge \mathrm{F}(c_0)$	$\{x_1 \vee x_2, 0\}$
$\mathsf{IV_1}$	3	$(\overline{x_1} \leftrightarrow \overline{x_2}\,\overline{x_3}) \wedge (\overline{x_2} \vee \overline{x_3} \to \overline{x_4}) \wedge \mathrm{T}(c_1)$	$\{x_1 \vee x_2, 1\}$
$\mathsf{IV_2}$	3	$(\overline{x_1} \leftrightarrow \overline{x_2}\,\overline{x_3}) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)$	$\{x_1 \vee x_2\}$
IE	2	$(x_1 \leftrightarrow x_2 x_3) \wedge (x_2 \vee x_3 \to x_4)$	$\{x_1 \wedge x_2, 0, 1\}$
$\mathsf{IE_0}$	3	$(x_1 \leftrightarrow x_2 x_3) \wedge (x_2 \vee x_3 \to x_4) \wedge \mathrm{F}(c_0)$	$\{x_1 \wedge x_2, 0\}$
$\mathsf{IE_1}$	2	$(x_1 \leftrightarrow x_2 x_3) \wedge \mathrm{T}(c_1)$	$\{x_1 \wedge x_2, 1\}$
$\mathsf{IE_2}$	3	$(x_1 \leftrightarrow x_2 x_3) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)$	$\{x_1 \wedge x_2\}$
IN	2	$\mathrm{EVEN}^4(x_1, x_2, x_3, x_4) \wedge x_1 x_4 \leftrightarrow x_2 x_3$	$\{\overline{x_1}, 0, 1\}$
$\mathsf{IN_2}$	3	$\mathrm{EVEN}^4_{4\neq}(x_1,\ldots,x_8) \wedge x_1 x_4 \leftrightarrow x_2 x_3$	$\{\overline{x_1}\}$
II	2	$(x_1 \leftrightarrow x_2 x_3) \wedge (\overline{x_4} \leftrightarrow \overline{x_2}\,\overline{x_3})$	$\{\mathrm{id}, 0, 1\}$
$\mathsf{II_0}$	2	$(\overline{x_1} \vee \overline{x_2}) \wedge (\overline{x_1}\,\overline{x_2} \leftrightarrow \overline{x_3}) \wedge \mathrm{F}(c_0)$	$\{\mathrm{id}, 0\}$
$\mathsf{II_1}$	2	$(x_1 \vee x_2) \wedge (x_1 x_2 \leftrightarrow x_3) \wedge \mathrm{T}(c_1)$	$\{\mathrm{id}, 1\}$
BR	3	$R^{1/3}_{3\neq}(x_1,\ldots,x_6) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)$	$\{\mathrm{id}\}$

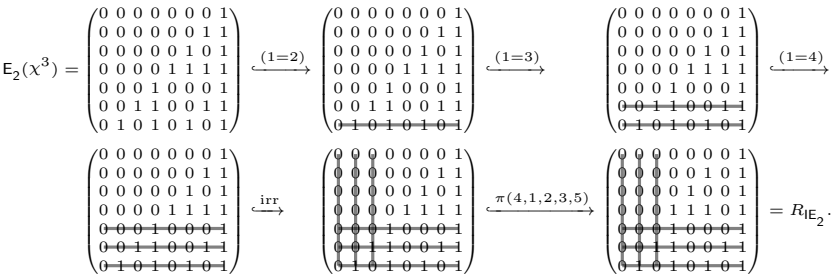

Fig. 2. Reduction sequence for R_{IE_2}. A black line indicates that the column/row has been removed from the relation.

For the remaining co-clones the proof is divided into two parts. First, we prove that the weak base for every co-clone IC in IM_0, IM_1, IM_2, ID_2, IV_1, IV_2, IE_0 and IE_2, can be obtained by collapsing columns from $\mathsf{C}(\chi^s)$. Second, we prove that for every $n \geq 2$ there exists simple weak bases for the co-clones $\mathsf{IS}^{\mathsf{n}}_0$, $\mathsf{IS}^{\mathsf{n}}_{02}$, $\mathsf{IS}^{\mathsf{n}}_{01}$, $\mathsf{IS}^{\mathsf{n}}_{00}$ and their duals $\mathsf{IS}^{\mathsf{n}}_1$, $\mathsf{IS}^{\mathsf{n}}_{12}$, $\mathsf{IS}^{\mathsf{n}}_{11}$, $\mathsf{IS}^{\mathsf{n}}_{10}$. To make the proofs more concise we introduce some admissible operations on relations which preserve the weak base property. Let R be an n-ary relation. Each rule $R \overset{*}{\hookrightarrow} R'$ implies that $\langle R' \rangle_{\not\exists} \subseteq \langle R \rangle_{\not\exists}$.

- $R \xhookrightarrow{(i=j)} R'$, $1 \leq i < j \leq n$,
  (Identify argument i with argument j),
- $R \xhookrightarrow{\pi(i_1,\ldots,i_n)} R'$, where π is the permutation $\pi(j) = i_j, 1 \leq j \leq n$, $1 \leq i_j \leq n$,
  (Swap arguments according to π),
- $R \xhookrightarrow{\text{irr}} R'$,
  (R' is irredundant).

Lemma 8. *Let* IC *be a co-clone,* R *an* n*-ary weak base for* IC*, and let* R' *be a relation such that* $R \overset{*}{\hookrightarrow} R'$ *for some rule* $\overset{*}{\hookrightarrow}$*. If* R' *is a base of* IC *then it is also a weak base of* IC*.*

Proof. We prove that $\langle R \rangle_{\not\exists} = \langle R' \rangle_{\not\exists}$ which implies that $\mathcal{I}_{\cup}(\mathrm{Pol(R)}) = \mathcal{I}_{\cup}(\mathrm{Pol(R')})$ and that R' is a weak base for IC. The first inclusion $\langle R \rangle_{\not\exists} \subseteq \langle R' \rangle_{\not\exists}$ is obvious since R is a weak base by assumption. To prove that $\langle R' \rangle_{\not\exists} \subseteq \langle R \rangle_{\not\exists}$ we show that $R' \in \langle R \rangle_{\not\exists}$ by giving a q.p.p. implementation of R' with R. There are three cases to consider. Either $R \xhookrightarrow{(i=j)}$

R', $1 \leq i < j \leq n$, in which case R' is the n-ary relation defined as $R'(x_1, \ldots, x_n) \equiv \mathrm{Eq}(\mathrm{x_i}, \mathrm{x_j}) \wedge \mathrm{R}(\mathrm{x_1}, \ldots, \mathrm{x_n})$, or $R \xhookrightarrow{\pi(i_1,\ldots,i_n)} R'$, in which case $R'(x_1, \ldots, x_n) \equiv R(x_{\pi(1)}, \ldots, x_{\pi(n)})$. The case when R' is irredundant follows easily since it can be obtained from R by removing all duplicate arguments.

Lemma 9. *The bases for* IM_0, IM_1, IM_2, ID_2, IV_1, IV_2, IE_0 *and* IE_2 *in Table 1 are minimal weak bases.*

Proof. We consider each case in turn. For every co-clone IC we write R_{IC} for the weak base from Table 1. We begin with the derivation for R_{IE_2}. From Table 1 we see that the core-size of IE_2 is 3, which means that we need to begin with the relation $\chi^3 = \{(0,0,0,0,1,1,1,1), (0,0,1,1,0,0,1,1), (0,1,0,1,0,1,0,1)\}$. The base of E_2 is according to Table 1 the binary function $f(x_1, x_2) = x_1 \wedge x_2$. When calculating $\mathsf{E}_2(\chi^3)$ we thus need to close χ^3 under f, which we do by repeatedly applying f to the tuples of χ^3 until no new tuples can be obtained, which means that the resulting relation is closed under f. In the case of $\mathsf{E}_2(\chi^3)$ we obtain four new tuples and get the relation $\mathsf{E}_2(\chi^3) = \chi^3 \cup \{$(0,0,0,0,1,1,1,1),(0,0,0,1,0,0,0,1), $(0,0,1,1,0,0,1,1)$, $(0,1,0,1,0,1,0,1\}$. By identifying arguments, removing redundant arguments and permutating arguments it is then possible to derive R_{IE_2}. The sequence is visualized in Figure 3. The remaining reductions are included below, where R, R', R'', ..., denotes intermediate relations.

$\mathsf{R}_0(\chi^1) \xhookrightarrow{(1=2)} R \xhookrightarrow{\mathrm{irr}} R_{\mathsf{IR}_0}$.

$\mathsf{R}_1(\chi^1) \xhookrightarrow{(1=2)} R \xhookrightarrow{\mathrm{irr}} R_{\mathsf{IR}_1}$.

$\mathsf{M}_0(\chi^2) \xhookrightarrow{(1=2)} R \xhookrightarrow{\mathrm{irr}} R' \xhookrightarrow{\pi(3,1,2)} R_{\mathsf{IM}_0}$.

$\mathsf{M}_1(\chi^2) \xhookrightarrow{(1=2)} R \xhookrightarrow{\mathrm{irr}} R_{\mathsf{IM}_1}$.

$\mathsf{M}_2(\chi^3) \xhookrightarrow{(1=2)} R \xhookrightarrow{(2=3)} R' \xhookrightarrow{(3=4)} R'' \xhookrightarrow{(5=6)} R''' \xhookrightarrow{\mathrm{irr}} R'''' \xhookrightarrow{\pi(3,1,2,4)} R_{\mathsf{IM}_2}$.

$\mathsf{D}_2(\chi^3) \xhookrightarrow{(1=2)} R \xhookrightarrow{\mathrm{irr}} R' \xhookrightarrow{\pi(5,4,1,3,2,6)} R_{\mathsf{ID}_2}$.

$\mathsf{V}_1(\chi^3) \xhookrightarrow{(4=8)} R \xhookrightarrow{(2=6)} R' \xhookrightarrow{(6=8)} R'' \xhookrightarrow{\mathrm{irr}} R''' \xhookrightarrow{\pi(4,2,3,1,5)} R_{\mathsf{IV}_1}$.

$\mathsf{V}_2(\chi^3) \xhookrightarrow{(4=8)} R \xhookrightarrow{(2=6)} R' \xhookrightarrow{(2=8)} R'' \xhookrightarrow{\mathrm{irr}} R''' \xhookrightarrow{\pi(4,2,3,1,5)} R_{\mathsf{IV}_2}$.

$\mathsf{E}_0(\chi^3) \xhookrightarrow{(1=2)} R \xhookrightarrow{(2=3)} R' \xhookrightarrow{(3=4)} R'' \xhookrightarrow{\mathrm{irr}} R''' \xhookrightarrow{\pi(5,1,2,3,4)} R_{\mathsf{IE}_0}$.

It is not hard to see that every relation R_{IC} is a base of IC. As in the previous cases the minimality of each weak base can be verified through

exhaustive search. As an example again consider

$$R_{\mathsf{IE}_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

First note that R_{IE_2} has no redundant or fictitious arguments. As for subset minimality removing three rows results in a relation in IR_2 while removing two rows from R_{IE_2} results in a relation in ID_1. Removing the first row results in a relation which generates BR and is hence no longer included in IE_2, removing the second or third row gives a relation in IM_2, and removing the fourth row gives a relation in IS^2_{10}. These properties can efficiently be tested using the algorithm in Creignou et al. [5]. Hence there is no relation $R' \subset R_{\mathsf{IE}_2}$ such that $\langle R' \rangle = \mathsf{IE}_2$ by which it follows that R_{IE_2} is a minimal weak base.

We now turn our attention towards the infinite parts of Post's lattice. In the sequel we sometimes represent relations by formulas in conjunctive normal form. If $\mathbf{x} = x_1, \ldots, x_n$ we use $\varphi(\mathbf{x})$ to denote a formula with n free variables, and the formula φ is then interpreted as defining an n-ary relation where the tuples coincide with the satisfying assignments of φ. If $\varphi = C_1 \wedge \ldots \wedge C_m$ is a formula with m clauses we say that C_i is a *prime implicate* of φ if φ does not entail any proper subclause of C_i. A formula φ is said to be *prime* if all of its clauses are prime implicates. Obviously any finite Boolean relation is representable by a prime formula. If R is an n-ary Boolean relation we can therefore prove that $R \in \langle \Gamma \rangle_{\not\equiv}$ by showing that $R(x_1, \ldots, x_n)$ can be expressed as a conjunction $\varphi_1(\mathbf{y_1}) \wedge \ldots \wedge \varphi_k(\mathbf{y_k})$, where each $\mathbf{y_i}$ is a vector over $x_1, \ldots, x_n$ and each φ_i is a prime formula representation of a relation in Γ. This is advantageous since relations in $\mathsf{IS}^{\mathsf{n}}_0$, $\mathsf{IS}^{\mathsf{n}}_{02}$, $\mathsf{IS}^{\mathsf{n}}_{01}$, $\mathsf{IS}^{\mathsf{n}}_{00}$, $\mathsf{IS}^{\mathsf{n}}_1$, $\mathsf{IS}^{\mathsf{n}}_{12}$, $\mathsf{IS}^{\mathsf{n}}_{11}$ and $\mathsf{IS}^{\mathsf{n}}_{10}$ are representable by prime *implicative hitting set-bounded* (IHSB) formulas [5]. We let IHSB^n_+ be the set of formulas of the form $(x_1 \vee \ldots \vee x_m), 1 \leq m \leq n, (\neg x_1), (\neg x_1 \vee x_2)$, and dually for IHSB^n_-. To avoid repetition we only present the full proof for $\mathsf{IS}^{\mathsf{n}}_{00}$ since the other cases follow through similar arguments.

Lemma 10. *The relation* $R_{\mathsf{IS}^{\mathsf{n}}_{00}}(x_1, \ldots, x_n, x, c_0, c_1) \equiv \mathrm{OR}(x_1, \ldots, x_n) \wedge (x \rightarrow x_1 \cdots x_n) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)$ *is a minimal weak base of* $\mathsf{IS}^{\mathsf{n}}_{00}$.

Proof. Let Γ be a constraint language such that $\langle \Gamma \rangle = \mathsf{IS}^{\mathsf{n}}_{00}$. Since Γ is finite we can without loss of generality restrict the proof to a single relation R of arity $m > n$ defined to be the cartesian product of all relations in Γ. We must prove that $R_{\mathsf{IS}^{\mathsf{n}}_{00}} \in \langle R \rangle_{\not\equiv}$ and by Creignou et al. [5] we know that R can be expressed as an IHSB^n_+ formula $\varphi(y_1, \ldots, y_m)$. The strategy

is therefore to prove that $R_{\mathsf{IS}^{\mathsf{n}}_{\mathsf{00}}}$ can be expressed as a conjunction of φ formulas without introducing any existentially quantified variables.

We first implement $\mathrm{F}(c_0)$ with $\varphi(y_1, \ldots, y_m)$ by identifying every variable y_j occurring in a negative clause $(\neg y_j)$ to c_0. There must exist at least one negative unary clause in φ since otherwise $\langle R \rangle = \mathsf{IS}^{\mathsf{n}}_{\mathsf{01}}$. Then, for any implicative clause $(\neg y_j \vee c_0)$ which also entails $(\neg c_0 \vee y_j)$ we identify y_j with c_0. For any remaining clause we identify all unbound variables with c_1. Since there must exist at least one positive prime clause this correctly implements $\mathrm{T}(c_1)$. Let $\varphi_{\mathrm{F,T}}(c_0, c_1)$ denote the resulting formula.

There is at least one n-ary prime clause of the form $(y_{j_1} \vee \ldots \vee y_{j_n})$ in φ since $\langle R \rangle = \mathsf{IS}^{\mathsf{n}}_{\mathsf{00}}$. We can therefore implement $\mathrm{OR}(x_1, \ldots, x_n)$ with $\varphi(y_1, \ldots, y_m)$ by first identifying $y_{j_1}, \ldots, y_{j_n}$ and $x_1, \ldots, x_n$. Let the resulting formula be φ'. Note that φ' might still contain unbound variables. In the subsequent formula we use x_i and $x_{i'}$ to denote variables in $x_1, \ldots, x_n$ and $y_j, y_{j'}$ to denote variables in φ' distinct from $x_1, \ldots, x_n$. Hence we need to replace each y_i still occurring in φ' with some x_i, c_0,c_1 or x. For every implicative clause C there are then three cases to consider.

1. $C = (\neg x_i \vee x_{i'})$,
2. $C = (\neg x_i \vee y_j)$,
3. $C = (\neg y_j \vee x_i)$.

The first case is impossible since $(x_1 \vee \ldots \vee x_n)$ was assumed to be prime. This also implies that the clauses $(\neg x_i \vee y_j)$ and $(\neg y_j \vee x_{i'})$ cannot occur simultaneously in the formula. For the second case we identify y_j with c_1. For the third case we identify y_j with c_0. If both the second and third case occur simultaneously we identify y_j with x_i. For any remaining clause we identify each unbound y_j with either c_0 or c_1. Thus the resulting formula $\varphi_{\mathrm{OR}}(x_1, \ldots, x_n, c_0, c_1)$ implements $\mathrm{OR}(x_1, \ldots, x_n)$.

In order to implement $(\neg x(\vee x_1 \wedge \ldots \wedge x_n))$ we need to ensure that $\neg x \vee x_i$ for all $1 \leq i \leq n$. Since $\langle R \rangle = \mathsf{IS}^{\mathsf{n}}_{\mathsf{00}}$ its prime formula representation $\varphi(y_1, \ldots, y_m)$ must contain a prime clause of the form $(\neg y_{j_1} \vee y_{j_2})$ where φ does not entail $(\neg y_{j_2} \vee y_{j_1})$. To implement $(\neg x \vee x_i)$ we therefore identify y_{j_1} with x and y_{j_2} with x_i. In the subsequent formula there are three implicative cases to consider:

1. $C = (\neg x \vee y_j)$,
2. $C = (\neg x_i \vee y_j)$,
3. $C = (\neg y_j \vee x_i)$,

where y_j denotes a variable distinct from x_i and x. In the first case we identify y_j with x_i, in the second case we identify y_j with c_1, and in the

third case we identify y_j with x. For any remaining positive clause we identify each unbound variable to c_1, and for any remaining negative unary clause $(\neg y_j)$ we identify y_j with c_0. In case there still exists an implicative clause $(\neg y_j \vee y_{j'})$ with two unbound variables y_j and $y_{j'}$ we identify y_j with c_0 and $y_{j'}$ with c_1. Let the resulting formula be $\varphi_{\rightarrow}(x, x_i, c_0, c_1)$. If we repeat the procedure for all $1 \leq i \leq n$ we see that $\varphi_{\rightarrow}(x, x_1, c_0, c_1) \wedge \ldots \wedge \varphi_{\rightarrow}(x, x_n, c_0, c_1)$ implements $(\neg x \vee x_1 \wedge \ldots \wedge x_n)$. Put together the formula $\varphi_{\mathrm{F,T}}(c_0, c_1) \wedge \varphi_{\mathrm{OR}}(x_1, \ldots, x_n, c_0, c_1) \wedge \varphi_{\rightarrow}(x, x_1, c_0, c_1) \wedge \ldots \wedge \varphi_{\rightarrow}(x, x_n, c_0, c_1)$ correctly implements $R_{\mathsf{IS}^{\mathsf{n}}_{00}}$ and furthermore only contains variables from $x_1, \ldots, x_n$, x, c_0, c_1. One can also prove that $R_{\mathsf{IS}^{\mathsf{n}}_{00}}$ is a base of $\mathsf{IS}^{\mathsf{n}}_{00}$ by giving an explicit p.p. definition of the base given by Böhler et al.[4]. As for minimality, first note that $R_{\mathsf{IS}^{\mathsf{n}}_{00}}$ does not contain any redundant or fictitious arguments and a case study similar to that of the preceding proof shows that removing any number of tuples from $R_{\mathsf{IS}^{\mathsf{n}}_{00}}$ results in a relation which either (1) does not contain an n-ary prime clause $(x_1 \vee \ldots \vee x_n)$ or (2) does not contain an implicative prime clause or (3) can no longer be expressed as an IHSB^n_+ formula.

Due to the duality of $\mathsf{IS}^{\mathsf{n}}_{0}, \mathsf{IS}^{\mathsf{n}}_{02}, \mathsf{IS}^{\mathsf{n}}_{01}, \mathsf{IS}^{\mathsf{n}}_{00}$ with $\mathsf{IS}^{\mathsf{n}}_{1}, \mathsf{IS}^{\mathsf{n}}_{12}, \mathsf{IS}^{\mathsf{n}}_{11}, \mathsf{IS}^{\mathsf{n}}_{10}$ we skip the latter proofs (which are almost identical since every relation can be written by a restricted IHSB^n_- formula) and instead refer to Lemma 10. Combining Lemmas 9 and 10 we have thus proved the main result of the paper.

Theorem 11. *The relations in Table 1 are minimal weak bases.*

4 Conclusions and future work

We have determined minimal weak bases for all Boolean co-clones with a finite base. Below are some topics worthy of future investigations.

The lattice of strong partial clones. Since the weak and plain base of a co-clone IC generates the smallest and largest elements of $\mathcal{I}(\mathsf{IC})$ it would be interesting to determine the full structure of this interval. Especially one would like to determine whether these intervals are finite, countably infinite or equal to the continuum.

Complexity of constraint problems. Each weak base effectively determines the constraint problem with the lowest complexity in a given co-clone. Example applications which follow from the categorization in this article include the NP-hard $\mathrm{CSP}(\cdot)$ problem in Jonsson et al. [8], with the property that it is solvable at least as fast as any other NP-hard Boolean $\mathrm{CSP}(\cdot)$. Are there other problems where similar classifications can be obtained?

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