On the Interval of Boolean Strong Partial Clones Containing Only Projections as Total Operations

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Abstract

A *strong partial clone* is a set of partial operations closed under composition and containing all partial projections. Let $\mathcal{X}$ be the set of all Boolean strong partial clones whose total operations are the projections. This set is of practical interest since it induces a partial order on the complexity of NP-complete *constraint satisfaction problems*. In this paper we study $\mathcal{X}$ from the algebraic point of view, and prove that there exists two intervals in $\mathcal{X}$, corresponding to natural constraint satisfaction problems, such that one is at least countably infinite and the other has the cardinality of the continuum.

1 Introduction

A k-ary *polymorphism* of a relation R is a homomorphism from the kth power of R to R. It is well-known that the set of all polymorphisms of a set of relations Γ form a *clone*, that is a set of operations closed under composition

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and containing all projections. The notion of a clone easily generalizes to partial operations, and it is known that the set of all *partial polymorphisms* of a set of relations Γ, $\mathrm{pPol}(\Gamma)$, form a *strong partial clone*, which is a set of partial operations closed under composition and containing all partial projections. Clone theory is not only a well-studied topic in universal algebra, but has many practical applications in computational complexity. One such example is the *constraint satisfaction problem* over a set of relations Γ ($\mathrm{CSP}(\Gamma)$), which can be viewed as the problem of deciding whether an existentially quantified conjunctive formula over Γ admits at least one model. For example, the NP-complete problem 1-in-3-SAT can be seen as a CSP problem over the ternary relation $R_{1/3} = \{(0,0,1),(0,1,0),(1,0,0)\}$. It has been proven that the polymorphisms of Γ determine the complexity of the CSP problem over Γ up to polynomial-time many-one reductions [5], while the partial polymorphisms of Γ can be used to study the complexity of the CSP problem over Γ with respect to stronger notions of reductions [6, 14]. More specifically, Jonsson et al. [6] proved that if $\mathrm{pPol}(\Gamma) \subseteq \mathrm{pPol}(\Delta)$ then $\mathrm{CSP}(\Delta)$ is solvable in $O(c^n)$ time whenever $\mathrm{CSP}(\Gamma)$ is solvable in $O(c^n)$ time (if Γ and Δ are both finite sets of relations). Hence, the partial polymorphisms of finite sets of relations induce a partial order on the complexity of NP-complete CSPs. Moreover, if Γ is a Boolean set of relations such that each operation in $\mathrm{Pol}(\Gamma)$ is idempotent, then $\mathrm{CSP}(\Gamma)$ is NP-complete if and only if $\mathrm{Pol}(\Gamma)$ consists only of projections [11, 13]. Put together, this implies that the set of strong partial clones containing only projections as total operations is a particularly interesting object of study, due to its relationship with NP-complete CSP problems. With these observation Jonsson et al. [6] then proved that there exists a relation $R_{1/3}^{\neq\neq\neq 01}$ such that (1) the CSP problem over $R_{1/3}^{\neq\neq\neq 01}$ is NP-complete but (2) there does not exist any NP-complete Boolean CSP problem with a strictly lower worst-case time-complexity. It is worth noting that the relation $R_{1/3}^{\neq\neq\neq 01}$ is also interesting from a purely algebraic point of view since it is known that $\mathrm{pPol}(R_{1/3}^{\neq\neq\neq 01})$ is the largest set of Boolean partial operations which does not contain any total operations (except the projections) [6, 14]. Jonsson et al. [6] then conjectured that the set of strong partial clones between $\mathrm{pPol}(R_{1/3})$ and $\mathrm{pPol}(R_{1/3}^{\neq\neq\neq 01})$ had a particularly simple structure consisting of only five elements. This conjecture turned out to be incorrect, and Lagerkvist & Roy [8] proved that the cardinality of this set is at least countably infinite.

In this paper we continue the investigation of $\mathrm{pPol}(R_{1/3}^{\neq\neq\neq 01})$, with a particular focus on the strong partial clones between $\mathrm{pPol}(R_{1/3})$ and $\mathrm{pPol}(R_{1/3}^{\neq\neq\neq 01})$. After having introduced the basic notions in Section 2, we begin (in Section 3.2) by proving that there exists at least countably many strong partial

clones between $\mathrm{pPol}(R_{1/3})$ and $\mathrm{pPol}(R^{01}_{1/3})$, where $R^{01}_{1/3} = R_{1/3} \times \{(0)\} \times \{(1)\}$. To prove this, we find a class of relations σ^k with the property that $\mathrm{pPol}(R_{1/3}) \subset \mathrm{pPol}(\{\sigma^k, R^{01}_{1/3}\}) \subset \mathrm{pPol}(R^{01}_{1/3})$ and $\mathrm{pPol}(\{\sigma^k, R^{01}_{1/3}\}) \subset \mathrm{pPol}(\{\sigma^{k+1}, R^{01}_{1/3}\})$, for each $k \geq 4$. In Section 3.3 we study the strong partial clones between $\mathrm{pPol}(R^{01}_{1/3})$ and $\mathrm{pPol}(R^{\neq\neq\neq 01}_{1/3})$, and prove that this set is of continuum cardinality, and therefore strengthen the results from Lagerkvist & Roy [8].

2 Preliminaries

In this section we introduce the basic terminology that will be used throughout the paper.

2.1 Operations and Relations

Let X be a finite and non-empty set of values. Without loss of generality we may assume that $X \subset \mathbb{N}$. In the particular case when $|X| = 2$ we assume that $X = \{0, 1\}$ and denote this set by $\mathbb{B}$. A k-ary *operation* over X, $k \geq 1$, is a function $X^k \to X$. We write OP_X for the set consisting of all operations over the set X. We will sometimes denote Boolean operations by their defining logical formulas, and for example write $\overline{x}$ for the operation $1 - x$, and we use $x \oplus y$ to denote addition modulo 2. A k-ary *partial operation* over X is a map $D \to X$ where $D \subseteq X^k$, and we write PAR_X for the set of all partial operations over X. The set D is called the *domain* of the partial operation f and we let $\mathrm{dom}(f) = D$. For both total and partial operations we let $\mathrm{ar}(f)$ denote the arity of f. If f and g are two k-ary partial operations g is said to be a *suboperation* of f if $\mathrm{dom}(g) \subseteq \mathrm{dom}(f)$ and $f(x) = g(x)$ for all $x \in \mathrm{dom}(g)$. We write $f_{|D'}$ for the suboperation obtained by restricting f to a set $D' \subseteq \mathrm{dom}(f)$.

A k-ary *relation* over X is a subset of X^k. The set of all relations over X is denoted by Rel_X. For $R_1, R_2 \in \mathrm{Rel}_X$ of arity n and m, we let $R_1 \times R_2 = \{(x_1, \ldots, x_{n+m}) \mid (x_1, \ldots, x_n) \in R_1, (x_{n+1}, \ldots, x_{n+m}) \in R_2\}$ denote their Cartesian product. Given a k-ary tuple $t = (x_1, \ldots, x_k) \in X^k$ we let $t[i] = x_i$ denote the ith component of t, and we let $\mathrm{Proj}_{i_1,\ldots,i_{k'}}(t) = (x_{i_1}, \ldots, x_{i_{k'}})$, $1 \leq k' \leq k$, denote the *projection* of t on the coordinates $i_1, \ldots, i_{k'} \in \{1, \ldots, k\}$. Similarly, if R is a k-ary relation we let $\mathrm{Proj}_{i_1,\ldots,i_{k'}}(R) = \{\mathrm{Proj}_{i_1,\ldots,i_{k'}}(t) \mid t \in R\}$. As a notational shorthand we let $\vec{0}^k$ and $\vec{1}^k$ denote the k-ary tuples $(0, \ldots, 0)$ and $(1, \ldots, 1)$.

It will be convenient to view a sequence of tuples $t_1, \ldots, t_n$ from a k-ary

relation R as a $k \times n$ matrix where the ith column consists of the tuple t_i. We let

$$\mathit{Rows}(t_1, \ldots, t_n) = ((t_1[1], \ldots, t_n[1]), \ldots, (t_n[k], \ldots, t_n[k]))$$

denote the tuple consisting of all rows of this matrix. When the exact ordering is not important we instead write

$$\mathit{RowSet}(t_1, \ldots, t_n) = \{(t_1[1], \ldots, t_n[1]), \ldots, (t_n[k], \ldots, t_n[k])\}$$

to denote the set consisting of all rows. For example, if $t_1 = (0,0,0), t_2 = (0,1,0)$, and $t_3 = (1,1,0)$ then

$$\mathit{Rows}(t_1, t_2, t_3) = ((0,0,1),(0,1,1),(0,0,1))$$

while

$$\mathit{RowSet}(t_1, t_2, t_3) = \{(0,0,1),(0,1,1)\}.$$

We will sometimes represent a relation as the set of models of a first-order formula, and if Γ is a set of relations we use the notation $R(x_1, \ldots, x_k) \equiv \varphi(x_1, \ldots, x_k)$, where $\varphi(x_1, \ldots, x_k)$ is a first-order formula over Γ with the free variables $x_1, \ldots, x_k$, to define the relation

$$R = \{(f(x_1), \ldots, f(x_k)) \mid f \text{ is a model of } \varphi(x_1, \ldots, x_k)\}.$$

2.2 Clones

Let Π_X be the set of all projections over $X \subseteq \mathbb{N}$, i.e. all operations π_i^n of the form $\pi_i^n(x_1, \ldots, x_i, \ldots, x_n) = x_i$. A *clone* over X is a set of operations $\mathsf{C} \subseteq \mathrm{OP}_X$ such that (1) $\mathsf{C} \supseteq \Pi_X$ and (2) C is closed under functional composition. More formally, the latter condition means that if $f, g_1, \ldots, g_m \in \mathsf{C}$, where f has arity m and the functions $g_1, \ldots, g_m$ all have the same arity n, then the composition $f \circ (g_1, \ldots, g_m)(x_1, \ldots, x_n) = f(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$ for all $x_1, \ldots, x_n \in X$, is included in C. For a set of total operations F we let $[F]$ be the smallest clone containing F, and call F a *basis* of $[F]$. A clone C is said to be *finitely generated* if there exists a finite set F such that $[F] = \mathsf{C}$, and is said to be *infinitely generated* otherwise. Every Boolean clone is known to be finitely generated, and the lattice of Boolean clones, *Post's lattice*, is countably infinite [11].

2.3 Strong Partial Clones

A *partial projection* is a suboperation of a projection. Let Π^p_X be the set of all partial projections over a set X. A set of partial operations C is a *strong partial clone* if (1) $\mathsf{C} \supseteq \Pi^p_X$ and (2) C is closed under composition of partial operations. It is well-known that this definition implies that C is closed under taking suboperations [12]. Composition of partial operations is defined analogously to the total case, i.e., if $f, g_1, \ldots, g_m \in \mathrm{PAR}_X$, $\mathrm{ar}(f) = m$, and $\mathrm{ar}(g_1) = \ldots = \mathrm{ar}(g_m) = n$, then $f \circ (g_1, \ldots, g_m)(x_1, \ldots, x_n) = f(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$, and the resulting function is defined for every tuple

$$(x_1, \ldots, x_n) \in \bigcap_{i=1}^{m} \mathrm{dom}(g_i)$$

such that

$$(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n)) \in \mathrm{dom}(f).$$

For a set of partial operations F we let $[F]_s$ be the smallest strong partial clone containing F, and similar to the total case the set F is said to be a *basis* of $[F]_s$. A strong partial clone C is said to be finitely generated if there exists a finite set of partial operations F such that $[F]_s = \mathsf{C}$, and is infinitely generated otherwise.

2.4 Galois Connections

It is known that clones and strong partial clones admit relational representations. As a shorthand we let

$$f(t_1, \ldots, t_n) = (f(t_1[1], \ldots, t_n[1]), \ldots, f(t_1[k], \ldots, t_n[k]))$$

and say that a relation R is *closed* or *invariant* under f if $f(t_1, \ldots, t_n) \in R$ for every sequence $t_1, \ldots, t_n \in R$. We also say that f is a *polymorphism* of R and let $\mathrm{Pol}(R)$ denote the set of all polymorphisms of the relation R. A relation R with $\mathrm{Pol}(R) = \Pi_X$ is said to be *strongly rigid*. Similarly, for a set of relations Γ we let $\mathrm{Pol}(\Gamma) = \bigcap_{R \in \Gamma} \mathrm{Pol}(R)$, and it is readily verified that f is a polymorphism of Γ if it preserves every relation in Γ. These notions can be generalized to partial operations, and we say that the n-ary partial operation f is a *partial polymorphism* of a k-ary relation R if $f(t_1, \ldots, t_n) \in R$ for every sequence $t_1, \ldots, t_n \in R$ such that $(t_1[1], \ldots, t_n[1]), \ldots, (t_1[k], \ldots, t_n[k]) \in \mathrm{dom}(f)$. We then let $\mathrm{pPol}(R)$ denote the set of all partial polymorphisms of the relation R, and let $\mathrm{pPol}(\Gamma)$ denote the set of all partial polymorphisms of the set of relations Γ. It is not difficult

to verify that sets of the form $\mathrm{Pol}(\Gamma)$ and $\mathrm{pPol}(\Gamma)$ form clones and strong partial clones, respectively. Moreover, if we let $\mathrm{Inv}(F)$ be the set of all invariant relations under the set of (partial) operations F, it is well-known (cf. Chapter 2.9 in Lau [10]) that $\mathrm{Pol}(\cdot)$ and $\mathrm{Inv}(\cdot)$ give rise to a *Galois* connection.

Theorem 1 ([1, 2, 4, 12])**.** *Let* Γ *and* Γ' *be two sets of relations. Then* $\Gamma \subseteq \mathrm{Inv}(\mathrm{Pol}(\Gamma'))$ *if and only if* $\mathrm{Pol}(\Gamma') \subseteq \mathrm{Pol}(\Gamma)$.

An analogous result is known for $\mathrm{Inv}(\cdot)$ and $\mathrm{pPol}(\cdot)$, due to Geiger [4] and Romov [12].

Theorem 2 ([4, 12])**.** *Let* Γ *and* Γ' *be two sets of relations. Then* $\Gamma \subseteq \mathrm{Inv}(\mathrm{pPol}(\Gamma'))$ *if and only if* $\mathrm{pPol}(\Gamma') \subseteq \mathrm{pPol}(\Gamma)$.

As a shorthand we let $\langle\Gamma\rangle_{\not\exists} = \mathrm{Inv}(\mathrm{pPol}(\Gamma))$. It is known that $\langle\Gamma\rangle_{\not\exists}$ is a set of relations closed under *quantifier-free primitive positive definitions*, (qfpp-definitions) over Γ, i.e., logical formulas of the form

$$\varphi(x_1, \ldots, x_n) \equiv R_1(\mathbf{x_1}) \wedge \ldots \wedge R_m(\mathbf{x_m})$$

where each $R_i \in \Gamma \cup \{\mathrm{Eq}\}$ and each $\mathbf{x_i}$ is a tuple of variables over $x_1, \ldots, x_n$. Here, Eq denotes the equality relation $\{(x, x) \mid x \in X\}$ over X. Hence, we can prove that $\mathrm{pPol}(\Gamma) \subseteq \mathrm{pPol}(\Gamma')$ by proving that each relation in Γ' admits a qfpp-definition over Γ.

2.5 Weak Bases

It turned out that the lattice of partial clones on a set with at least two elements is very complex, and significant efforts were made by several authors to study parts of it. The question of describing the general position of the lattice of all total clones within the lattice of partial clones was raised by D. Lau. For example, given a total clone C, determine cardinality of the set of the partial clones on X whose total component is C. For $\mathbb{B}$, this problem was completely solved in [3] and the corresponding problem for strong partial clones was solved in [15]. We need the following definition:

Definition 3. *Let* C *be a clone over* X*. We let*

$$\mathrm{Int}(\mathsf{C}) = \{\mathrm{pPol}(\Gamma) \mid \Gamma \subseteq \mathrm{Rel}_X, \mathrm{Pol}(\Gamma) = \mathsf{C}\}.$$

We will sometimes be concerned with even more fine-grained sets of strong partial clones and, given two strong partial clones C_1 and C_2, write $\mathrm{Int}(\mathsf{C}_1, \mathsf{C}_2)$ to denote the set

$$\{\mathrm{pPol}(\Gamma) \mid \Gamma \subseteq \mathrm{Rel}_X, \mathsf{C}_1 \subseteq \mathrm{pPol}(\Gamma) \subseteq \mathsf{C}_2\}.$$

It is worth noting that the smallest element in the set $\mathrm{Int}(\mathsf{C})$ is simply $[\mathsf{C}]_s$, the strong partial clone obtained by closing C under suboperations, but it is also known that each such set admits a largest element.

Theorem 4. *[14] Let C be a finitely generated clone. Then*

$$\bigcup_{\mathrm{pPol}(\Gamma)\in\mathrm{Int}(\mathsf{C})} \mathrm{pPol}(\Gamma) \in \mathrm{Int}(\mathsf{C}).$$

It is also known that whenever $\mathrm{Inv}(\mathsf{C})$ is finitely generated, there exists a relation R over X such that

$$\mathrm{pPol}(R) = \bigcup_{\mathrm{pPol}(\Gamma)\in\mathrm{Int}(\mathsf{C})} \mathrm{pPol}(\Gamma).$$

Using the terminology of Schnoor and Schnoor [14] we call a relation satisfying this condition a *weak basis* of $\mathrm{Inv}(\mathsf{C})$. Moreover, for the Boolean domain, all weak bases have been fully described [7]. In this paper we have a particular interest in the weak basis

$$R_{1/3}^{\neq\neq\neq 01} = \{(0,0,1,1,1,0,0,1),(0,1,0,1,0,1,0,1),(1,0,0,0,1,1,0,1)\}$$

of $\mathrm{Rel}_{\mathbb{B}}$. Note that $\mathrm{Proj}_{1,2,3}(R_{1/3}^{\neq\neq\neq 01}) = R_{1/3}$, that the third, fourth, and fifth arguments are complement of the three first, and that the two last arguments are constant 0 and 1, respectively. As remarked in Section 1, $\mathrm{pPol}(R_{1/3}^{\neq\neq\neq 01})$ can be viewed as the set of all Boolean partial operations f such that $[\{f\}]_s \cap \mathrm{OP}_{\mathbb{B}} = \Pi_{\mathbb{B}}$. Hence, it is the largest Boolean strong partial clone whose only total operations are the projections. Somewhat surprisingly, it is known that this strong partial clone is not finitely generated, in contrast to the strong partial clone consisting of all Boolean partial operations (which can be generated by a partial Sheffer operation).

Theorem 5. *[9] Let Γ be a finite set of relations over a finite domain such that $\mathrm{Pol}(\Gamma)$ has a basis consisting of unary operations. Then $\mathrm{pPol}(\Gamma)$ is infinitely generated.*

Since $\mathrm{Pol}(\{R_{1/3}^{\neq\neq\neq 01}\}) = \Pi_{\mathbb{B}}$, it can be generated by a single projection, and so by Theorem 5 we have that $\mathrm{pPol}(R_{1/3}^{\neq\neq\neq 01})$ is infinitely generated.

3 The Structure Between pPol($R_{1/3}$) and pPol($R_{1/3}^{\neq\neq\neq 01}$)

Jonsson et al. [6] conjectured that Int(pPol($R_{1/3}$), pPol($R_{1/3}^{\neq\neq\neq 01}$)) consisted of pPol($R_{1/3}$), pPol($R_{1/3}^{01}$), pPol($R_{1/3}^{\neq 01}$), pPol($R_{1/3}^{\neq\neq 01}$), and pPol($R_{1/3}^{\neq\neq\neq 01}$), where $R_{1/3}^{\neq\neq 01} = \mathrm{Proj}_{1,2,3,4,5,7,8}(R_{1/3}^{\neq\neq\neq 01})$, $R_{1/3}^{\neq 01} = \mathrm{Proj}_{1,2,3,4,6,7}(R_{1/3}^{\neq\neq 01})$, and $R_{1/3}^{01} = R_{1/3} \times R^0 \times R^1$. This was disproven by Lagerkvist & Roy who proved that Int(pPol($R_{1/3}^{01}$), pPol($R_{1/3}^{\neq\neq\neq 01}$)) is at least countably infinite [8]. However, the questions of whether pPol($R_{1/3}$) is covered by pPol($R_{1/3}^{01}$), and whether Int(pPol($R_{1/3}^{01}$), pPol($R_{1/3}^{\neq\neq\neq 01}$)) is in fact of continuum cardinality, were left open. After introducing some simplifying notation in Section 3.1, we answer both of these questions in Section 3.2 and Section 3.3. Before reading these sections in greater detail, the reader may first consult Figure 1 for a visualization of the main results.

3.1 Partial Polymorphisms of Strongly Rigid Relations

In order to describe the partial polymorphisms of strongly rigid relations we make the following definition.

Definition 6. *Let R be a relation and f a k-ary partial operation. We let*

$$\begin{aligned}\mathrm{Cover}_R(\mathrm{dom}(f)) = \{\, &\mathit{RowSet}(t_1, \ldots, t_k) \mid t_1, \ldots, t_k \in R, \\ &\mathit{RowSet}(t_1, \ldots, t_k) \subseteq \mathrm{dom}(f)\}.\end{aligned}$$

This immediately gives rise to the following lemma.

Lemma 7. *Let R be a strongly rigid relation and f a partial operation over a finite set X. If $f_{|C} \in \Pi_X^p$ for every $C \in \mathrm{Cover}_R(\mathrm{dom}(f))$ then $f \in \mathrm{pPol}(R)$.*

For $R_{1/3}$ and related relations the following concept can be used to characterize $\mathrm{Cover}_{R_{1/3}}(\mathrm{dom}(f))$. For $k, k' \geq 1$, a set $\{\omega_1, \ldots, \omega_{k'}\} \subseteq \mathbb{B}^k$ is an *exact k'-cover* if $\omega_1[i] + \ldots + \omega_{k'}[i] = 1$ for every $i \in \{1, \ldots, k\}$.

3.2 pPol($R_{1/3}$) and pPol($R_{1/3}^{01}$)

In this section we study the structure of the set Int(pPol($R_{1/3}$), pPol($R_{1/3}^{01}$)) and prove that it is at least countably infinite. First we define a useful class of relations.

Definition 8. *For each $k \geq 4$ let σ^k denote the relation*

$$\begin{aligned}&\{(x_1, \ldots, x_k, \overline{x_1}, \ldots, \overline{x_k}, y_1, \ldots, y_{k-3}, \overline{y_1}, \ldots, \overline{y_{k-3}}, 0) \mid \\ &\Sigma_{i=1}^k x_i = 1, y_1 + x_1 + x_2 = 1, \ldots, y_{k-3} + \Sigma_{i=1}^{k-2} x_i = 1\}.\end{aligned}$$

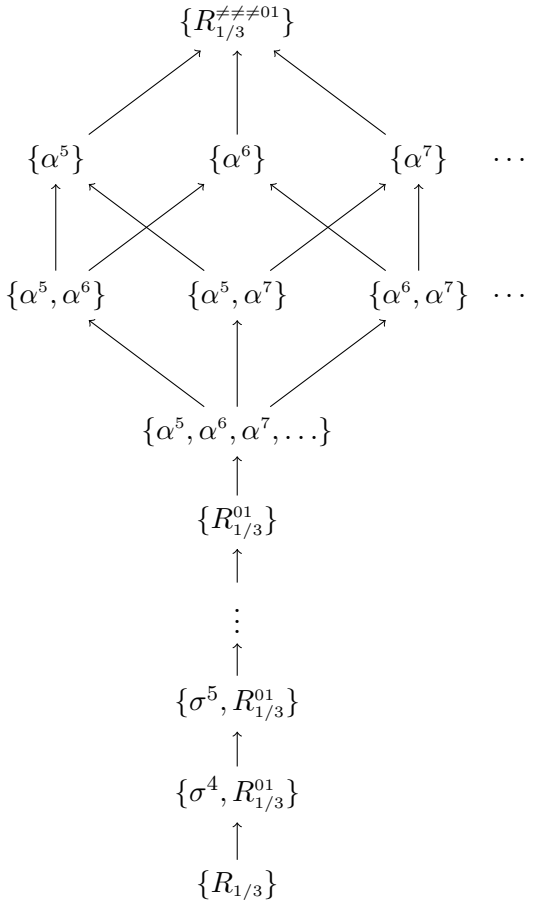

Figure 1: A visualization of $\mathrm{Int}(\mathrm{pPol}(R_{1/3}), \mathrm{pPol}(R_{1/3}^{\neq\neq\neq 01}))$. A directed arrow from Γ to Δ means that $\mathrm{pPol}(\Gamma) \subset \mathrm{pPol}(\Delta)$. Some inclusions are omitted.

Hence, if $t_1, \ldots, t_{k'} \in \sigma^k$ then $RowSet(t_1, \ldots, t_{k'})$ contains an exact k'-cover, the complement of these tuples, $2(k-3)$ tuples determined by the exact k'-cover, and a constant 0 tuple. For example, the relation σ^5 may be visualized as

$$\sigma^5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We first claim that pPol($R_{1/3}$) $\subseteq$ pPol(σ^4). To see this, note that σ^4 can be qfpp-defined as

$$\begin{aligned} &\sigma^4(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, y_1, z_1, v_0) \equiv \\ &\quad R_{1/3}(x_1, x_2, y_1) \wedge R_{1/3}(x_3, x_4, z_1) \wedge R_{1/3}(y_1, z_1, v_0) \wedge \\ &\quad \bigwedge_{i=1}^{4} (x_i, x_{i+4}, v_0). \end{aligned}$$

It is then not difficult to see that the partial operation $f(0,0,1) = f(0,1,0) = f(1,0,0) = 0$ preserves σ^4 and $R_{1/3}^{01}$ but not $R_{1/3}$. Since pPol($R_{1/3}$) $\subset$ pPol($R_{1/3}^{01}$) it therefore follows that pPol($R_{1/3}$) $\subset$ pPol($\{\sigma^4, R_{1/3}^{01}\}$). More generally it also holds that pPol($\{\sigma^k, R_{1/3}^{01}\}$) $\subset$ pPol($\{\sigma^{k+1}, R_{1/3}^{01}\}$).

Lemma 9. pPol($\{\sigma^k, R_{1/3}^{01}\}$) $\supset$ pPol($\{\sigma^{k-1}, R_{1/3}^{01}\}$) *for each* $k \geq 5$.

Proof. To prove the inclusion pPol($\{\sigma^k, R_{1/3}^{01}\}$) $\supseteq$ pPol($\{\sigma^{k-1}, R_{1/3}^{01}\}$) we show that $\sigma^k \in \langle\{\sigma^{k-1}\}\rangle_{\not\exists}$, via the qfpp-definition

$$\begin{aligned} &\sigma^k(x_1, \ldots, x_{2k}, y_1, \ldots, y_{k-3}, z_1, \ldots, z_{k-3}, v_0) \equiv \\ &\quad \sigma^{k-1}(x_1, \ldots, x_{k-2}, y_{k-3}, x_{k+1}, \ldots, x_{2k-2}, z_{k-3}, y_1, \ldots, \\ &\quad y_{k-4}, z_1, \ldots, z_{k-4}, v_0) \wedge \sigma^{k-1}(x_k, x_{k-1}, \ldots, x_3, z_1, x_{2k}, \\ &\quad x_{2k-1}, \ldots, x_{k+3}, y_1, z_{k-3}, z_{k-4}, \ldots, z_1, y_{k-4}, \ldots, y_1, v_0). \end{aligned}$$

For the proper inclusion, define the $(k-1)$-ary partial operation f such that $\mathrm{dom}(f) = \mathit{RowSet}(\sigma^{k-1})$, and let $f(\vec{0}^{k-1}) = 1$ and $f(x) = \pi_1^{k-1}(x)$ otherwise. By definition f does not preserve σ^{k-1}, and it directly follows from Theorem 10 in Lagerkvist & Roy that f preserves $R_{1/3}^{01}$ [8]. Hence, all that remains is to prove that f preserves σ^k, which we do by showing that $\mathrm{Cover}_{\sigma^k}(\mathrm{dom}(f)) = \emptyset$. Assume there exists $t_1, \ldots, t_{k-1} \in \sigma^k$ such that $\mathit{RowSet}(t_1, \ldots, t_{k-1}) \subseteq \mathrm{dom}(f)$. Since $\Sigma_{j=1}^{k} t_i[j] = 1$ for each $i \in \{1, \ldots, k-1\}$ and $k > k-1$, there exists $l \in \{1, \ldots, k\}$ such that $t_1[l] = \ldots = t_{k-1}[l] = 0$. But due to Definition 8, $t_1[l+k] = \ldots = t_k[l+k] = 1$, implying that $\mathit{RowSet}(t_1, \ldots, t_{k-1}) \not\subseteq \mathrm{Cover}_{\sigma^k}(\mathrm{dom}(f))$ since $\vec{1}^{k-1} \notin \mathrm{dom}(f)$. □

We have thus proved the following.

Theorem 10. *The set* $\mathrm{Int}(\mathrm{pPol}(R_{1/3}), \mathrm{pPol}(R_{1/3}^{01}))$ *is at least countably infinite.*

3.3 pPol($R_{1/3}^{01}$) and pPol($R_{1/3}^{\neq\neq\neq 01}$)

Following Lagerkvist & Roy [8] we define the following class of relations.

Definition 11. *Let* $k \geq 5$. *The relation* α^k *is defined as*

$$\begin{aligned}\alpha^k(&x_1, \ldots, x_k, y_1, \ldots, y_{k-3}, z_1, \ldots, z_{k-3}, w_1, \ldots, w_{k-4},\\ &v_0, v_1)\\ &\equiv \exists x_{k+1}, \ldots, x_{2k}. \bigwedge_{i=3}^{k-2} R_{1/3}(x_1, x_i, w_{i-2}) \wedge R^1(v_1) \wedge\\ &\sigma^k(x_1, \ldots, x_{2k}, y_1, \ldots, y_{k-3}, z_1, \ldots, z_{k-3}, v_0).\end{aligned}$$

For each $k \geq 5$, let $\alpha_1, \ldots, \alpha_k$ be an enumeration of the tuples in α^k such that $\alpha_1[k] = 1$, $\alpha_2[k-1] = 1$, …, $\alpha_k[1] = 1$. Under this enumeration of tuples we for example see that the matrix corresponding to α^6 can be

visualized as:

$$\alpha^6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Let $\mathcal{V}_k = \mathit{Rows}(\alpha_1, \ldots, \alpha_k)[2(k-2) + 2(k-3)]$, and define the k-ary partial operation f^k as $\mathrm{dom}(f^k) = \mathit{RowSet}(\alpha_1, \ldots, \alpha_k)$ and $f^k(x) = \pi_1^k(x)$ for every $x \in \mathrm{dom}(f^k) \setminus \{\mathcal{V}_k\}$ and $f^k(\mathcal{V}_k) = \alpha_1[2(k-2) + 2(k-3)] \oplus 1$. For example, we have that $\mathcal{V}_6 = (1, 1, 0, 1, 1, 0)$ and therefore that $f(\mathcal{V}_6) = 0$. We then have the following result from Lagerkvist & Roy [8].

Theorem 12 (Lagerkvist & Roy [8])**.** *Let $k \geq 5$. Then (1)* $\mathrm{pPol}(R_{1/3}^{01}) \subset \mathrm{pPol}(\alpha^k) \subset \mathrm{pPol}(R_{1/3}^{\neq\neq\neq 01})$ *and (2) $f^k \notin \mathrm{pPol}(\alpha^k)$ but $f^k \in \mathrm{pPol}(\alpha^{k'})$ for every $5 \leq k < k'$.*

We will shortly see that f^k preserves $\alpha^{k'}$ whenever $k' \geq 5$ and $k' \neq k$. First we state the following lemma, whose proof is trivial due to the fact that f^k is defined as a projection on the first argument whenever its arguments are distinct from $\mathcal{V}_k$.

Lemma 13. *Let $k, k' \geq 5$. Then $f^k_{|C} \in \Pi^p_{\mathbb{B}}$ for every $C \in \mathrm{Cover}_{\alpha^{k'}}(\mathrm{dom}(f^k))$ such that $\mathcal{V}_k \notin C$.*

Before the proof of the following lemma we make a few observations regarding the domain of the function f^k. Due to Definition 11 this set consists of

1. k tuples $x_1, \ldots, x_k$ such that $\{x_1, \ldots, x_k\}$ forms an exact k-cover,
2. $k-3$ tuples $y_1, \ldots, y_{k-3}$ such that $\{x_1, \ldots, x_i, y_{i-1}\}$, $2 \leq i \leq k-2$, forms an exact k-cover,

3. $k-3$ tuples $\overline{y_1}, \ldots, \overline{y_{k-3}}$ such that $\{x_k, \ldots, x_{i+1}, \overline{y_{i-1}}\}$, $2 \leq i \leq k-2$, forms an exact k-cover,

4. $k-4$ tuples $z_1, \ldots, z_{k-4}$ such that $\{x_1, x_i, z_{i-2}\}$ forms an exact k-cover for $3 \leq i \leq k-2$, and

5. two constant tuples $\vec{0}^k$ and $\vec{1}^k$.

It is also worth remarking that each y_i satisfies the condition that $y_i[j] = 1$ if $1 \leq j \leq k-i+1$ and $y_i[j] = 0$ if $k-i+1 < j \leq k$, and that $z_i[k] = 0$, $z_i[i+2] = 0$, and $z_i[j] = 1$ otherwise. In particular, $\mathrm{dom}(f^k)$ contains the tuple $\mathcal{V}_k$, which satisfies $\mathcal{V}_k[3] = 0$ and $\mathcal{V}_k[k] = 0$, and $\mathcal{V}_k[i] = 1$ otherwise. We are now ready to prove the main result of this section.

Lemma 14. *Let $k \geq 5$. Then $f^k \notin \mathrm{pPol}(\alpha^k)$ but $f^k \in \mathrm{pPol}(\alpha^{k'})$ for every $k' \geq 5$ such that $k' \neq k$.*

Proof. Assume first that $k' > k$. Then the result follows from Theorem 12. Hence, in the forthcoming proof, assume that $k' < k$. We will prove that whenever $C \in \mathrm{Cover}_{\alpha^{k'}}(\mathrm{dom}(f^k))$ then $\mathcal{V}_k \notin C$, which by combining Lemma 7 and Lemma 13 proves that $f^k \in \mathrm{pPol}(\alpha^{k'})$. Hence, for $t_1, \ldots, t_k \in \alpha^{k'}$ let

$$(a_1, \ldots, a_{k'}, b_1, \ldots, b_{k'-3}, c_1, \ldots, c_{k'-3}, d_1, \ldots, d_{k'-4}, \vec{0}^k, \vec{1}^k)$$

denote the tuples in $\mathit{Rows}(t_1, \ldots, t_k)$. There are now a few cases to consider. However, if $\mathcal{V}_k = a_i$, $1 \leq i \leq k'$, or if $\mathcal{V}_k = b_i$, $\mathcal{V}_k = c_i$ for $1 \leq i \leq k'-3$, then the proof follows the same argument as Lemma 13 in Lagerkvist & Roy [8].

Therefore, assume there exists some $i \in \{1, \ldots, k'-4\}$ such that $d_i = \mathcal{V}_k$. This implies that $\{a_1, a_{i+2}, d_i\}$ forms an exact k-cover, and we may without loss of generality assume that $\vec{0}^k \notin \{a_1, a_{i+2}, d_i\}$ as otherwise $\overline{\mathcal{V}_k} \in \{a_1, a_{i+2}, d_i\}$, which is impossible since $\overline{\mathcal{V}_k} \notin \mathrm{dom}(f^k)$. Note in particular that this implies that a_1 and a_{i+2} both contain exactly one entry equal to 1. Assume without loss of generality that $a_1[k] = 1$ and that $a_{i+2}[3] = 1$. We will first prove that $\Sigma a_j = 1$ for any $j \in \{1, \ldots, k'-2\}$, where $\Sigma a_j = a_j[1] + \ldots + a_j[k]$.

First, assume that $a_j = \vec{0}^k$ for some $j \in \{1, \ldots, k'-2\}$. The case when $j = 1$ cannot occur since we have already established that $\Sigma a_1 = 1$, and $j = 2$ is impossible since it implies that $b_1 = \overline{a_1} \notin \mathrm{dom}(f)$, due to the fact that $\{a_1, a_2, b_1\}$ must be an exact k-cover. Similarly, we see that $d_{j-2} = \overline{a_1} \notin \mathrm{dom}(f)$ if $\Sigma a_j = 0$ for $j \in \{3, \ldots, k'-2\}$, since $\{a_1, a_j, d_{j-2}\}$ must be an exact k-cover.

Second, assume that $\Sigma a_j > 1$ for some $1 \leq j \leq k'-2$. Since the cases when $j = 1$ or $j = i+2$ are impossible we assume that $j \neq 1$ and

that $j \neq i+2$. Since $a_1 = (0,0,\ldots,0,1)$ and $a_{i-2} = (0,0,1,0,\ldots,0)$, this implies that $a_j = (1,1,0,\ldots,0)$, as otherwise $a_j \notin \mathrm{dom}(f^k)$. But then, since $\{a_1, a_j, d_{j-2}\}$ must form an exact k-cover, f^k cannot be defined on d_{j-2}.

Hence, $\Sigma a_j = 1$ for each $1 \leq j \leq k'-2$, but since $k' < k$ this implies that $\Sigma a_{k'-1} > 1$ or that $\Sigma a_{k'} > 1$. We assume that $\Sigma a_{k'-1} > 1$ since the case when $\Sigma a_{k'} > 1$ is entirely symmetric. Due to the assumption that $a_{k'-1} \in \mathrm{dom}(f^k)$, $a_1 = (0,\ldots,0,1)$ and $a_{i-2} = (0,0,1,0,\ldots,0)$, this in fact implies that $a_{k'-1} = (1,1,0,\ldots,0)$. Now, since $\{a_{k'-1}, a_{k'}, c_{k'-2}\}$ must form an exact k-cover, either $a_{k'} = \vec{0}^k$ or $c_{k'-2} = \vec{0}^k$, as otherwise $a_{k'} \notin \mathrm{dom}(f^k)$ or $c_{k'-2} \notin \mathrm{dom}(f^k)$. However, the latter case is impossible since it implies that $a_{k'-1} = \overline{a_{k'}}$, which cannot happen since $\{a_1,\ldots,a_{k'}\}$ is an exact k-cover, and since $\Sigma a_j = 1$ for $1 \leq j \leq k'-2$. Hence, $a_{k'} = \vec{0}^k$, and $c_{k'-2} = \overline{a_{k'}}$. Now, since $\Sigma a_j = 1$ for every $1 \leq j \leq k'-2$, it follows that $|\{t_3,\ldots,t_k\}| = k-2$, i.e., the tuples $t_3,\ldots,t_k$ do not contain any repetitions. However, since $t_1 \notin \{t_3,\ldots,t_k\}$, and there cannot exist t_j such that $t_j[k'] = 1$, there are only $k'-2$ tuples to choose from. This is impossible since $k' < k$, and we conclude that $\mathcal{V}_k \notin \mathit{RowSet}(t_1,\ldots,t_k)$. □

Theorem 15. *The set* $\mathrm{Int}(\mathrm{pPol}(R^{01}_{1/3}), \mathrm{pPol}(R^{\neq\neq\neq 01}_{1/3}))$ *is of continuum cardinality.*

Proof. Let $\mathbb{N}_{\geq 5} = \mathbb{N} \setminus \{0,1,2,3,4\}$. We will prove that there exists an injective function h from $\{X \mid X \subseteq \mathbb{N}_{\geq 5}\}$ to $\mathrm{Int}(\mathrm{pPol}(R^{01}_{1/3}), \mathrm{pPol}(R^{\neq\neq\neq 01}_{1/3}))$. Define $h(X) = \mathrm{pPol}(\{\alpha^i \mid i \in X\})$ for every $X \subseteq \mathbb{N}_{\geq 5}$. We claim that $\mathrm{pPol}(\{\alpha^i \mid i \in X\}) \neq \mathrm{pPol}(\{\alpha^i \mid i \in Y\})$ whenever $X, Y \subseteq \mathbb{N}_{\geq 5}$ and $X \neq Y$. Indeed, if $X \neq Y$ then there exist $i_1 \in X$ or $i_2 \in Y$ such that $i_1 \notin Y$ or $i_2 \notin X$. But then $f_{i_1} \notin \mathrm{pPol}(\{\alpha^i \mid i \in X\})$ and $f_{i_1} \in \mathrm{pPol}(\{\alpha^i \mid i \in Y\})$, or $f_{i_2} \notin \mathrm{pPol}(\{\alpha^i \mid i \in X\})$ and $f_{i_2} \in \mathrm{pPol}(\{\alpha^i \mid i \in Y\})$, due to Lemma 14. □

4 Concluding Remarks

We have studied the set $\mathrm{Int}(\Pi_{\mathbb{B}})$, with a particular focus on describing the strong partial clones between $\mathrm{pPol}(R_{1/3})$ and $\mathrm{pPol}(R^{\neq\neq\neq 01}_{1/3})$. By generalizing the results from Lagerkvist & Roy [8] we have proven that $\mathrm{Int}(\mathrm{pPol}(R_{1/3}), \mathrm{pPol}(R^{01}_{1/3}))$ is at least countably infinite and that $\mathrm{Int}(\mathrm{pPol}(R^{01}_{1/3}), \mathrm{pPol}(R^{\neq\neq\neq 01}_{1/3}))$ is of continuum cardinality. These results open up a few distinct directions for future research. First, note that Lemma 9 raises the possibility that $\mathrm{pPol}(\{R^{01}_{1/3}, \sigma^k\})$ is *covered* by $\mathrm{pPol}(\{R^{01}_{1/3}, \sigma^{k+1}\})$, i.e., that there does not exist any strong partial clone inbetween. If this is indeed the case then the cardinality of $\mathrm{Int}(\mathrm{pPol}(R_{1/3}), \mathrm{pPol}(R^{01}_{1/3}))$ might be countably infinite. Second, it would be

interesting to determine the strong partial clones covered by $\mathrm{pPol}(R_{1/3}^{\neq\neq\neq 01})$. Since $\mathrm{pPol}(R_{1/3}^{\neq\neq\neq 01})$ is infinitely generated (by Theorem 5) a good starting point is to determine whether every strong partial subclone of $\mathrm{pPol}(R_{1/3}^{\neq\neq\neq 01})$ is contained in a maximal strong partial subclone of $\mathrm{pPol}(R_{1/3}^{\neq\neq\neq 01})$, or if there exists a countably infinite chain of strong partial clones in $\mathrm{Int}(\Pi_{\mathbb{B}})$ such that the union of this chain equals $\mathrm{pPol}(R_{1/3}^{\neq\neq\neq 01})$.

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