Fine-Grained Complexity of Constraint Satisfaction Problems through Partial Polymorphisms: A Survey

Dedicated to the memory of Professor Ivo Rosenberg.

Miguel Couceiro*[1], Lucien Haddad†[2], and Victor Lagerkvist‡[3]

[1]Université de Lorraine, CNRS, Inria Nancy G.E., LORIA, F-54000 Nancy, France
[2]Department of Mathematics & Computer Science, Royal Military College of Canada, Kingston, Ontario, Canada
[3]Department of Computer and Information Science, Linköping University, Linköping, Sweden

Abstract

Constraint satisfaction problems (CSPs) are combinatorial problems with strong ties to universal algebra and clone theory. The recently proved CSP dichotomy theorem states that finite-domain CSPs are always either tractable or NP-complete. However, among the intractable cases there is a seemingly large variance in complexity, which cannot be explained by the classical algebraic approach using polymorphisms. In this contribution we will survey an alternative approach based on partial polymorphisms, which is useful for studying the fine-grained complexity of NP-complete CSPs. Moreover, we will state some challenging open problems in the research field.

1 Algebraic Background

Let $k \geq 2$ be an integer and let $\mathbf{k}$ be a k-element set. Without loss of generality we assume that $\mathbf{k} := \{0, \ldots, k-1\}$. An n-ary *relation* R over $\mathbf{k}$ is a subset of $\mathbf{k}^n$, and we write $\mathrm{ar}(R) = n$ to denote its arity, and $\mathrm{Rel}_{\mathbf{k}}$ for the set of all relations over $\mathbf{k}$. For a positive integer n, an *n-ary partial operation* on $\mathbf{k}$ is a map $f : \mathrm{dom}(f) \to \mathbf{k}$ where $\mathrm{dom}(f)$ is a subset of $\mathbf{k}^n$, called the *domain* of f. Let $\mathrm{Par}^{(n)}(\mathbf{k})$ denote the set of all n-ary partial operations on $\mathbf{k}$ and let $\mathrm{Par}(\mathbf{k}) := \bigcup_{n\geq 1} \mathrm{Par}^{(n)}(\mathbf{k})$. An n-ary partial operation g is said to be a *total operation* if $\mathrm{dom}(g) = \mathbf{k}^n$, and we let $\mathrm{Op}^{(n)}(\mathbf{k})$ be the set of all n-ary total operations on $\mathbf{k}$ and $\mathrm{Op}(\mathbf{k}) := \bigcup_{n\geq 1} \mathrm{Op}^{(n)}(\mathbf{k})$.

*miguel.couceiro@{loria,Inria}.fr
†haddad-l@rmc.ca
‡victor.lagerkvist@liu.se

For every positive integer n and each $1 \leq i \leq n$, let e_i^n denote the *n-ary i-th projection* defined by $e_i^n(a_1, \ldots, a_n) = a_i$ for all $(a_1, \ldots, a_n) \in \mathbf{k}^n$. Furthermore, let $J_{\mathbf{k}} := \{e_i^n \mid 1 \leq i \leq n, n \in \mathbb{N} \setminus \{0\}\}$ be the set of all (total) projections. Partial operations on $\mathbf{k}$ are composed in a natural way. For additional details we refer the reader to Lau [31].

Definition 1. *A clone is a composition closed subset of* $\mathrm{Op}(\mathbf{k})$ *containing* $J_{\mathbf{k}}$*, and a partial clone on* $\mathbf{k}$ *is a composition closed subset of* $\mathrm{Par}(\mathbf{k})$ *containing* $J_{\mathbf{k}}$*. A partial clone is said to be strong if it is closed under taking suboperations*[1]*.*

It is well known that a partial clone C is strong if and only if $\mathrm{Str}(J_{\mathbf{k}}) \subseteq C$ (see, e.g., Lemma 2.11 in Haddad and Börner [8]). Moreover, strong partial clones are exactly those partial clones that are *determined by relations* in the following way. Let $h, n \geq 1$ be integers, and let R be an h-ary relation on $\mathbf{k}$. An n-ary partial operation f on $\mathbf{k}$ is said to *preserve* R if for every $h \times n$ matrix $M = [M_{ij}]$ whose columns $M_{*j} \in R$, and whose rows $M_{i*} \in \mathrm{dom}(f)$, the h-tuple $(f(M_{1*}), \ldots, f(M_{h*})) \in R$. Note that if there is no $h \times n$ matrix $M = [M_{ij}]$ whose columns $M_{*j} \in R$ and whose rows $M_{i*} \in \mathrm{dom}(f)$, then f preserves R. It is not difficult to see that

$$\mathrm{pPol}(R) := \{f \in \mathrm{Par}(\mathbf{k}) \mid f \text{ preserves } R\}$$

is a strong partial clone called the *partial clone determined by the relation* R. Similarly, if Γ is a set of relations over $\mathbf{k}$ we write $\mathrm{pPol}(\Gamma)$ for the set $\bigcap_{R \in \Gamma} \mathrm{pPol}(R)$. In the total case we similarly write $\mathrm{Pol}(R)$ for the set of total polymorphisms of R and $\mathrm{Pol}(\Gamma)$ if Γ is a set of relations. If $F \subseteq \mathrm{Par}(\mathbf{k})$ then we also write $\mathrm{Inv}(F)$ for the set of relations preserved by all partial operations in F. Sets of the form $\mathrm{Inv}(F)$ are known as *relational clones*, or *co-clones*, if each operation in F is total, otherwise they are called *weak systems* or *weak co-clones*.

The set of partial clones on $\mathbf{k}$ forms a lattice $\mathcal{L}_{P_{\mathbf{k}}}$ under inclusion, in which the infimum is the set-theoretical intersection. Similarly, the set of strong partial clones on $\mathbf{k}$ also forms a lattice $\mathcal{L}_{\mathrm{Str}(P_{\mathbf{k}})}$, which is a sublattice of $\mathcal{L}_{P_{\mathbf{k}}}$. By definition, J_k and $\mathrm{Str}(J_k)$ are the least elements of $\mathcal{L}_{P_{\mathbf{k}}}$ and $\mathcal{L}_{\mathrm{Str}(P_{\mathbf{k}})}$, respectively. For further background see, e.g., [8, 13, 15]. For $F \subseteq \mathrm{Par}(\mathbf{k})$, let $[F]_s$ denote the intersection of all strong partial clones on $\mathbf{k}$ containing F. Similarly, for $F \subseteq \mathrm{Op}(\mathbf{k})$, let $[F]$ be the intersection of all clones on $\mathbf{k}$ containing F, and in both cases we write $[f]$ or $[f]_s$ when $F = \{f\}$ is singleton.

2 Constraint Satisfaction Problems

In a *constraint satisfaction problem* (CSP) the objective is to assign values to variables subjected to a set of constraints deciding permissible assignments. It is typically phrased as the decision problem of determining if there exists an assignment respecting all constraints, and we begin with the following definition which is predominant in computer science literature.

[1]For $f, g \in \mathrm{Par}(\mathbf{k})$, g is a *suboperation of* f, $g \leq f$, if $g = f|_{\mathrm{dom}(g)}$. For $F \subseteq \mathrm{Par}(\mathbf{k})$, we denote its closure under taking suboperations by $\mathrm{Str}(F)$.

A *constraint satisfaction problem* (CSP) over a set $\mathbf{k}$ is defined as follows.
INSTANCE: A tuple (V, C) where V is a finite set, and C a finite set of the form (R_i, t_i) where $R_i \in \mathrm{Rel}_{\mathbf{k}}$ and $t_i \in V^{\mathrm{ar}(R_i)}$.
QUESTION: Does there exist a function $f\colon V \to \mathbf{k}$ such that $(f(x_i^1), \ldots, f(x_i^{\mathrm{ar}(R_i)})) \in R_i$ for each $(R_i, (x_i^1, \ldots, x_i^{\mathrm{ar}(R_i)})) \in C$?

The set $\mathbf{k}$ is called the *domain* of the CSP — not to be confused with the domain of a partial operation. If $k = 2$ then $\mathbf{k}$ is said to be *Boolean*. The members of V are referred to as *variables* and are usually denoted by x or v. A tuple $(R_i, t_i) \in C$ is called a *constraint*, and we typically write $R(t_i)$ instead of (R_i, t_i). The function f, if it exists, is called a *solution*, a *model*, or a *satisfying assignment*.

CSPs can be further specified by fixing a set of relations Γ, called a *constraint language*. This problem is then referred to as $\mathrm{CSP}(\Gamma)$ and is restricted to instances (V, C) where $R_i \in \Gamma$ for each constraint $(R_i, t_i) \in C$. If Γ is Boolean then $\mathrm{CSP}(\Gamma)$ can be viewed as a *satisfiability* problem, usually written $\mathrm{SAT}(\Gamma)$.

Observe that if we associate a constraint language Γ over a domain D with a relational signature τ, Γ can be treated as a relational structure Γ^τ. With this viewpoint an instance $\{\{v_1, \ldots, v_n\}, C\}$ of $\mathrm{CSP}(\Gamma)$ can be viewed as an existentially quantified τ-formula $\exists v_1, \ldots, v_n \colon \bigwedge_{(R_i, t_i) \in C} R_i(t_i)$, and the question is then simply to check whether this τ-formula admits at least one model. It is also possible to rephrase $\mathrm{CSP}(\Gamma)$ as a homomorphism problem, i.e., an instance I of $\mathrm{CSP}(\Gamma)$ can be seen as a τ-structure $\mathcal{I}$ and the goal is then to check if there exists a homomorphism between $\mathcal{I}$ and Γ^τ.

Example 1. *Let $R_{1/3} = \{(0,0,1), (0,1,0), (1,0,0)\}$. Then $\mathrm{SAT}(\{R_{1/3}\})$ can be seen as an alternative formulation of the monotone 1-in-3-SAT problem which is well-known to be NP-complete. By choosing a suitable Boolean Γ, a large range of satisfiability problems can be represented as a $\mathrm{CSP}(\Gamma)$ problem. For example, for each $k \geq 1$ let Γ^k_{SAT} be the set of relations of the form $\{0,1\}^k \setminus \{t\}$ for a single k-ary Boolean tuple t. Then $\mathrm{SAT}(\Gamma^k_{\mathrm{SAT}})$ can be verified to be an alternative formulation of k-SAT which is NP-complete for $k \geq 3$.*

Example 2. *Let us also consider a few non-Boolean examples. One of the prototypical examples of a CSP is the k-COLOURING problem: given an undirected graph (V, E), can (V, E) be coloured using at most k colours? To formulate this problem as a CSP we take the relation $R_{\neq_k} = \{(x, y) \in \mathbf{k}^2 \mid x \neq y\}$ and for each $(x, y) \in E$ introduce a constraint $R_{\neq_k}(x, y)$. It is also easy to find examples of tractable CSPs, i.e., CSPs solvable in polynomial time. One such example is systems of linear equations $x_1 + \ldots + x_n = 0 \,(\mathrm{mod}\, k)$ which can be solved in polynomial time using Gaussian elimination. As we will see in Section 3 this discrepancy in complexity between tractable and NP-complete CSPs can be explained using algebraic methods.*

Although this survey mainly focuses on finite-domain CSPs, a substantial amount of research is dedicated towards infinite-domain CSPs. This is especially true in artificial intelligence where many classical problems are intrinsically linked to constraints over infinite domains. Some examples include spatial and temporal reasoning problems such as ALLEN'S INTERVAL ALGEBRA, THE REGION CONNECTION CALCULUS, and the RECTANGLE ALGEBRA (cf. the surveys [4, 19]).

3 Polymorphisms and the Complexity of CSP

Feder & Vardi conjectured that $\mathrm{CSP}(\Gamma)$ is either tractable or NP-complete [20]; this conjecture is usually referred to as the CSP *dichotomy conjecture*. It was then realized that several classical algorithms resulting in tractability, e.g., Gaussian elimination and k-consistency, in a uniform manner could be explained by the presence of certain polymorphisms of Γ [24]. More generally, Jeavons proved the following reducibility result, usually interpreted as "the polymorphisms of Γ determine the complexity of $\mathrm{CSP}(\Gamma)$ up to polynomial-time reductions".

Theorem 2 ([23])**.** *Let Γ and Δ be two finite constraint languages over $\mathbf{k}$. If $\mathrm{Pol}(\Delta) \subseteq \mathrm{Pol}(\Gamma)$, then $\mathrm{CSP}(\Gamma)$ is polynomial-time many-one reducible to $\mathrm{CSP}(\Delta)$.*

Proof. The condition $\mathrm{Pol}(\Delta) \subseteq \mathrm{Pol}(\Gamma)$ is well-known to be equivalent to the condition $\Gamma \subseteq \mathrm{Inv}(\mathrm{Pol}(\Delta))$ [6, 7, 21]. This is furthermore known to imply that each $R \in \Gamma$ can be expressed as a conjunction of constraints over Δ, possibly with the introduction of additional variables and equality constraints, and the reduction then proceeds as a classical "gadget reduction" where each constraint is transformed into the corresponding constraints over Δ. ☐

To obtain a dichotomy for $\mathrm{CSP}(\Gamma)$ over $\mathbf{k}$ one would then need to fully describe all operations over $\mathbf{k}$ and determine all combinations resulting in tractable CSPs. However, such an undertaking turned out to be unecessary, due to the realization that the classical complexity of $\mathrm{CSP}(\Gamma)$ only depends on the *identities*, or the *Strong Maltsev conditions*, satisfied by the polymorphisms of Γ [11]. For example, if $\mathrm{Pol}(\Gamma)$ contains a *Maltsev operation* satisfying the identities $m(x, x, y) \approx y, m(x, y, y) \approx x$, then $\mathrm{CSP}(\Gamma)$ is tractable since it can be solved by the simple algorithm for Maltsev constraints [10]. The main advantage of this viewpoint is therefore that it suffices to describe all identities resulting in tractable CSPs rather than all concrete operations. This approach recently culminated in the following dichotomy theorem.

Theorem 3 ([9, 42])**.** *Let Γ be a constraint language over $\mathbf{k}$. Then $\mathrm{CSP}(\Gamma)$ is either tractable or NP-complete.*

For additional details concerning the classification project of CSP and the algebraic approach based on Strong Maltsev conditions, see e.g. the survey by Barto [1].

4 Partial Polymorphisms and the Fine-Grained Complexity of CSP

We begin this section by discussing the rather vague term "fine-grained complexity" in relationship to CSPs in Section 4.1, before we describe how the algebraic approach based on partial polymorphisms can be used to study this question in Section 4.2 and Section 4.3.

4.1 Fine-Grained Complexity

Recall from Section 3 that polymorphisms are useful for studying the classical complexity of CSPs up to polynomial-time reductions. However, there is reason to believe that even NP-complete problems can exhibit a striking difference in complexity, and that it may be disadvantageous to group them together under the guise of polynomial-time reductions. For example, $\mathrm{SAT}(\{R_{1/3}\})$ from Example 1, is known to be solvable in $O(1.0984^n)$ time, where n denotes the number of variables [41], whereas it is not known whether the unrestricted SAT problem $\mathrm{SAT}(\mathrm{Rel}_{\{0,1\}})$ is solvable in $O(c^n)$ time for $c < 2$. This phenomena is not restricted to CSPs: for example, van Rooij *et al.* [5] proved that the Partition Into Triangles problem restricted to graphs of maximum degree 4 can be solved in $O(1.0222^n)$ time despite being NP-complete.

Our main concern in this survey paper is thus to study the complexity of NP-complete CSPs with regards to $O(c^n)$ time complexity (although we will also mention related applications in Section 4.2). To make this question more precise we begin with the following definition.

Definition 4. *Let $k \geq 2$. For $\Gamma \subseteq \mathrm{Rel}_k$ we define $\mathsf{T}(\Gamma)$ as*

$$\mathsf{T}(\Gamma) = \inf\{c \mid \mathrm{CSP}(\Gamma) \textit{ is solvable in time } 2^{cn}\}$$

where n is the number of variables in an instance of $\mathrm{CSP}(\Gamma)$*.*

Note that it might be the case that $\mathrm{CSP}(\Gamma)$ is solvable in $O(2^{(c+\varepsilon)n})$ time for every $\varepsilon > 0$ despite not being solvable in $O(2^{cn})$ time — hence, the use of infimum in Definition 4 is necessary. It is important to observe that no concrete value of $\mathsf{T}(\Gamma)$ is known when $\mathrm{CSP}(\Gamma)$ is NP-complete. Before we begin studying the function T and its connection to partial polymorphisms we need to state the following conjecture, which is of central importance in current research on fine-grained complexity and lower bounds.

Definition 5. *The* exponential-time hypothesis *(ETH) [22] conjectures that* $\mathsf{T}(\Gamma^3_{\mathrm{SAT}}) > 0$*.*

In other words, the ETH states that there exists a $c > 0$ such that 3-SAT is not solvable in $O(2^{cn})$ time. Although not immediate from Definition 5, the ETH is also known to imply that the sequence $\mathsf{T}(\Gamma^3_{\mathrm{SAT}}), \mathsf{T}(\Gamma^4_{\mathrm{SAT}}), \ldots$ increases infinitely often, i.e., that for every k there exists $k' > k$ such that $\mathsf{T}(\Gamma^k_{\mathrm{SAT}}) < \mathsf{T}(\Gamma^{k'}_{\mathrm{SAT}})$ [22]. This makes it tempting to also conjecture that the limit of the sequence $\mathsf{T}(\Gamma^3_{\mathrm{SAT}}), \mathsf{T}(\Gamma^4_{\mathrm{SAT}}), \ldots$ equals 1; a conjecture known as the *strong exponential-time hypothesis* (SETH) [12, 22]. Hence, under this conjecture the unrestricted SAT problem cannot be solved in $O(2^{cn})$ time for *any* $c < 1$.

Let us also remark that $\mathrm{CSP}(\Gamma)$ for $\Gamma \subseteq \mathrm{Rel}_{\mathbf{k}}$ is always solvable in $O(k^n)$ time by simply enumerating all possible assignments over $\mathbf{k}$. Hence, $\mathsf{T}(\Gamma) \leq \log_2(k)$ for every $\Gamma \subseteq \mathrm{Rel}_{\mathbf{k}}$. It is also known that if $\Gamma \subset \mathrm{Rel}_{\mathbf{k}}$ is finite then $\mathrm{CSP}(\Gamma)$ is solvable in $O(c^n)$ time for a $c < k$ [40], implying that $\mathsf{T}(\Gamma) < \log_2(k)$.

4.2 Weak Bases

Schnoor & Schnoor [37] investigated connections between partial polymorphisms and the complexity of CSPs. However, their motivation was not to study the fine-grained complexity of CSPs, but to analyse the classical complexity of CSP-like problems incompatible with polymorphisms.

Example 3. $\mathrm{CSP}(\Gamma)$ *is sometimes said to be* a priori *compatible with polymorphisms due to the existence of Theorem 2. In contrast, there exist problems proven to be* a posteriori *compatible with polymorphisms, in the sense that* $\mathrm{Pol}(\Gamma)$ *determines whether the problem is tractable or intractable, but where an analogue of Theorem 2 cannot be proven. One such example is the problem of finding a surjective model of a* $\mathrm{SAT}(\Gamma)$ *instance (*$\mathrm{SUR\text{-}SAT}(\Gamma)$*), which is NP-complete if* $\mathrm{Pol}(\Gamma)$ *is essentially unary and tractable otherwise. Curiously, almost all CSP-like problems studied in the literature turn out to be either a priori or a posteriori compatible with polymorphisms, and only a handful of concrete counter examples exist, e.g., enumerating models of* $\mathrm{CSP}(\Gamma)$ *with polynomial delay [36], the inverse satisfiability problem over infinite constraint languages [29], and the maximum satisfiability problem [18].*

Problems that are not a priori compatible with polymorphisms may instead be compatible with partial polymorphisms. It is, for example, straightforward to prove that if $\mathrm{pPol}(\Gamma) \subseteq \mathrm{pPol}(\Delta)$ then $\mathrm{SUR\text{-}SAT}(\Delta)$ is polynomial-time many-one reducible to $\mathrm{SUR\text{-}SAT}(\Gamma)$. However, since the lattice of Boolean strong partial clones $\mathcal{L}_{\mathrm{Str}(P_{\{0,1\}})}$ is largerly unexplored, this is not as useful as one might first believe. Schnoor & Schnoor [37] realized that for many classification purposes, there is typically no need to consider the whole lattice $\mathcal{L}_{\mathrm{Str}(P_{\{0,1\}})}$, but only a small fragment corresponding to *weak bases*.

Definition 6. *[37] Let* $C = \mathrm{Pol}(\Gamma)$ *be a clone over* $\mathbf{k}$ *where* Γ *is finite. A set of relations* $\Gamma_w \subseteq \mathrm{Rel}_{\mathbf{k}}$ *is said to be a* weak base *of* $\mathrm{Inv}(C)$ *if (1)* $\mathrm{Pol}(\Gamma_w) = C$ *and (2)* $\mathrm{pPol}(\Delta) \subseteq \mathrm{pPol}(\Gamma_w)$ *for each set* $\Delta \subseteq \mathrm{Rel}_{\mathbf{k}}$ *such that* $\mathrm{Pol}(\Delta) = C$.

Example 4. *Let us again consider* $\mathrm{SUR\text{-}SAT}(\Gamma)$ *and assume that we are given a weak base* Γ_w *of a co-clone* $\mathrm{Inv}(C)$*. If we can prove that* $\mathrm{SUR\text{-}SAT}(\Gamma_w)$ *is NP-complete, then NP-completeness also carries over to every* Γ *such that* $\mathrm{Pol}(\Gamma) = C$*. Hence, equipped with a weak base of each Boolean co-clone, we in practice only need to consider Post's lattice [33] rather than* $\mathcal{L}_{\mathrm{Str}(P_{\{0,1\}})}$.

Schnoor & Schnoor [37] also described a procedure for constructing weak bases for co-clones satisfying the preconditions in Definition 6, which was leveraged by Lagerkvist to provide a list of weak bases for all Boolean co-clones [27]. We will not describe this method in detail, but remark that it is based on the observation that the algebra whose universe consists of all n-ary operations in C can be viewed as a relation R, with the property that any partial operation not preserving R can be extended to a total operation. In the literature, this construction has been referred to as a *n-generated free algebra* [1], or the *n-th graphic* [32]. Using a similar strategy as in Example 4, weak bases have been used to obtain complexity dichotomies for several Boolean CSP-like problems incompatible with polymorphisms [2, 3, 29, 37, 38].

4.3 An Algebraic Approach Based on Partial Polymorphisms

We are now ready to present the link between partial polymorphisms and the function T, allowing us to study the fine-grained complexity of CSPs using partial polymorphisms.

Theorem 7 ([25]). *Let* Γ *and* Δ *be two finite sets of relations. If* $\text{pPol}(\Gamma) \subseteq \text{pPol}(\Delta)$ *then* $\mathsf{T}(\Delta) \leq \mathsf{T}(\Gamma)$.

Proof. The proof is similar to Theorem 2: $\text{pPol}(\Gamma) \subseteq \text{pPol}(\Delta)$ is known to be equivalent to the condition $\Delta \subseteq \text{Inv}(\text{Pol}(\Gamma))$ [21, 34], which in turn implies that each $R \in \Delta$ can be written as a set of constraints over Γ without introducing any fresh variables. If each constraint in a CSP(Δ) instance is rewritten in this manner we obtain an instance of CSP(Γ) over the same set of variables, giving the bound $\mathsf{T}(\Delta) \leq \mathsf{T}(\Gamma)$. □

Now, let C be a clone such that $\text{Pol}(\Gamma) = C$ and CSP(Γ) is NP-complete. Theorem 7 then offers an algebraic method to analyse $\mathsf{T}(\Gamma)$ by studying the properties of $\mathcal{I}_{\text{Str}}(C) := \{\text{pPol}(\Gamma) \mid \text{Pol}(\Gamma) = C\}$. For example, if $\mathcal{I}_{\text{Str}}(C)$ is finite, then the fine-grained complexity of CSP(Γ) would fall into a finite number of cases. Hence, as a rough approximation, we would like to know the cardinality of $\mathcal{I}_{\text{Str}}(\text{Pol}(\Gamma))$ when CSP(Γ) is NP-complete. A dichotomy has been proved for Boolean clones, with the surprising implication that these sets are always either finite or equal to the continuum.

Theorem 8 ([17, 39]). *Let* C *be a Boolean clone. Then* $\mathcal{I}_{\text{Str}}(C)$ *is finite if* $C \supseteq \text{Pol}(\{\{(0,1),(1,0)\},\{(0,1)\}\})$ *or* $C \supseteq \text{Pol}(\{\{(0,0),(0,1),(1,1)\},\{(0,1)\}\})$, *and is of continuum cardinality otherwise.*

By inspecting Post's lattice of Boolean clones [33] one can then verify that the finite cases of Theorem 8 hold for only 10 clones. Furthermore, it is known that SAT(Γ) is NP-complete if and only if $\text{Pol}(\Gamma) = [f_{\neg}]$ or $\text{Pol}(\Gamma) = J_{\{0,1\}}$, where $f_{\neg}(x) = 1 - x$ [35], implying that $\mathcal{I}_{\text{Str}}(\text{Pol}(\Gamma))$ is of continuum cardinality whenever SAT(Γ) is NP-complete. Clearly, the fact that $\mathcal{I}_{\text{Str}}(\text{Pol}(\Gamma))$ is of continuum cardinality in these cases says very little of their actual complexity, but at least suggests that one needs a different line of attack than trying to obtain a characterization akin to Post's lattice.

Let us for the moment concentrate on Boolean constraint languages Γ such that $\text{Pol}(\Gamma) = J_{\{0,1\}}$, which subsume the examples 1-IN-3-SAT and k-SAT from Example 1. Even though fully describing $\mathcal{I}_{\text{Str}}(J_{\{0,1\}})$ does not appear straightforward due to Theorem 8, there are still plenty of questions relevant for the fine-grained complexity of SAT(Γ). For example, does $\mathcal{I}_{\text{Str}}(J_{\{0,1\}})$ admit a greatest element, and if this is the case, is it then possible to describe the maximal elements? Similarly, is it possible to describe the minimal strong partial clones of $\mathcal{I}_{\mathrm{Str}}(J_{\{0,1\}})$ — provided they exist (note that a unique least element trivially exists, namely $\mathrm{Str}(J_{\{0,1\}}))^2$. These questions are of interest in fine-grained complexity since we from Theorem 7 would expect that "small" members of $\mathcal{I}_{\mathrm{Str}}(J_{\{0,1\}})$ corresponds to SAT problems with high time complexity, and that "large" members of $\mathcal{I}_{\mathrm{Str}}(J_{\{0,1\}})$ give rise to SAT problems of low time complexity. In fact, recalling the concept of a weak base R of a co-clone $\mathrm{Inv}(C)$ from Section 4.2, one of these questions can be answered immediately: $\mathrm{pPol}(R) \supseteq \mathrm{pPol}(\Gamma)$ for each $\mathrm{pPol}(\Gamma) \in \mathcal{I}_{\mathrm{Str}}(J_{\{0,1\}})$ implies that $\mathrm{pPol}(R)$ is the greatest element. Furthermore, $\mathrm{Inv}(J_{\{0,1\}})$ is known to admit a particularly simple weak base $R^{\neq\neq\neq 01}_{1/3} = \{(0,0,1,1,1,0,0,1),(0,1,0,1,0,1,0,1),(1,0,0,0,1,1,0,1)\}$ [27]. This observation was then leveraged by Jonsson et al. [25] to show that $\mathrm{SAT}(\{R^{\neq\neq\neq 01}_{1/3}\})$ results in the "easiest NP-complete SAT problem", in the following sense.

Theorem 9 ([25])**.** $\mathrm{SAT}(\{R^{\neq\neq\neq 01}_{1/3}\})$ *is NP-complete and* $\mathsf{T}(\{R^{\neq\neq\neq 01}_{1/3}\}) \leq \mathsf{T}(\Gamma)$ *for* any *Boolean constraint language* Γ *such that* $\mathrm{SAT}(\Gamma)$ *is NP-complete.*

Proof. We give a short sketch of the most important ideas. If $\mathrm{SAT}(\Gamma)$ is NP-complete then by Schaefer's dichotomy theorem either $\mathrm{Pol}(\Gamma) = [f_{\neg}]$ or $\mathrm{Pol}(\Gamma) = J_{\{0,1\}}$ [35]. It is then known that the relation $R = R^{\neq\neq\neq 01}_{1/3} \cup \{(f_{\neg}(t) \mid t \in R^{\neq\neq\neq 01}_{1/3})\}$ is a weak base of $\mathrm{Inv}(\{f_{\neg}\})$ [27], and from Theorem 7 we then conclude that $\mathsf{T}(\{R\}) \leq \mathsf{T}(\Gamma)$ or $\mathsf{T}(\{R^{\neq\neq\neq 01}_{1/3}\}) \leq \mathsf{T}(\Gamma)$, since $\mathrm{pPol}(\Gamma) \subseteq \mathrm{pPol}(R)$ or $\mathrm{pPol}(\Gamma) \subseteq \mathrm{pPol}(R^{\neq\neq\neq 01}_{1/3})$. Hence, it is sufficient to prove that $\mathsf{T}(\{R^{\neq\neq\neq 01}_{1/3}\}) \leq \mathsf{T}(\{R\})$, which can be accomplished by a polynomial-time many-one reduction only introducing a constant number of fresh variables (see Lemma 19 in Jonsson el al. [25]). □

This result was later extended to a broad class of finite-domain CSPs, so-called *ultraconservative* CSPs, which can be viewed as $\mathrm{CSP}(\Gamma)$ problems where Γ contains all unary relations over the domain.

Theorem 10 ([26])**.** *For each* $\mathbf{k}$ *there exists a relation* $R_{\mathbf{k}} \in \mathrm{Rel}_{\mathbf{k}}$ *such that (1)* $\mathrm{CSP}(\{R_{\mathbf{k}}\})$ *is NP-complete, and (2)* $\mathsf{T}(\{R\}) \leq \mathsf{T}(\Gamma)$ *for* any *ultraconservative* $\Gamma \subseteq \mathrm{Rel}_{\mathbf{k}}$ *such that* $\mathrm{CSP}(\Gamma)$ *is NP-complete.*

Jonsson *et al.* [25] also conjectured that the strong partial clones between $\mathrm{pPol}(R_{1/3})$ and $\mathrm{pPol}(R^{\neq\neq\neq 01}_{1/3})$ had a simple structure consisting of only three elements $\mathrm{pPol}(R^{01}_{1/3})$, $\mathrm{pPol}(R^{\neq 01}_{1/3})$, $\mathrm{pPol}(R^{\neq\neq 01}_{1/3})$, such that $\mathrm{pPol}(R_{1/3}) \subset \mathrm{pPol}(R^{01}_{1/3}) \subset \mathrm{pPol}(R^{\neq 01}_{1/3}) \subset \mathrm{pPol}(R^{\neq\neq 01}_{1/3}) \subset \mathrm{pPol}(R^{\neq\neq\neq 01}_{1/3})$. However, this turned out to be false: Lagerkvist & Roy first showed the existence of countably many strong partial clones between $\mathrm{pPol}(R^{01}_{1/3})$ and $\mathrm{pPol}(R^{\neq 01}_{1/3})$, $\mathrm{pPol}(R^{\neq 01}_{1/3})$ and $\mathrm{pPol}(R^{\neq\neq 01}_{1/3})$, and $\mathrm{pPol}(R^{\neq\neq 01}_{1/3})$ and $\mathrm{pPol}(R^{\neq\neq\neq 01}_{1/3})$ [28]. This was later refined by Couceiro *et al.* [14] where it was proven that one can actually construct a family of strong partial clones of continuum size between each of these pairs.

[2] We follow the standard terminology where minimal/maximal clones are those directly above/below the greatest/least element in the clone lattice.

4.4 The Non-Existence of Minimal Strong Partial Clones

We now turn to the question of minimal strong partial clones in $\mathcal{I}_{\mathrm{Str}}(J_{\{0,1\}})$, i.e., $\mathrm{pPol}(\Gamma) \in \mathcal{I}_{\mathrm{Str}}(J_{\{0,1\}})$ such that $\mathrm{pPol}(\Gamma) \supset \mathrm{Str}(J_{\{0,1\}})$ but there does not exist $\mathrm{pPol}(\Delta) \in \mathcal{I}_{\mathrm{Str}}(J_{\{0,1\}})$ such that $\mathrm{pPol}(\Gamma) \supset \mathrm{pPol}(\Delta) \supset \mathrm{Str}(J_{\{0,1\}})$. The existence of a minimal element $\mathrm{pPol}(\Gamma)$ would have interesting consequences in the light of the SETH, in particular if if $\mathsf{T}(\Gamma) < 1$, since $\mathrm{SAT}(\Gamma)$ would then belong to the class of "hardest" NP-complete SAT problems which are still easier than the unrestricted SAT problem. However, this question has a surprisingly straightforward resolution, as proven by Couceiro et al. [16]: there are no minimal strong partial clones. More specifically, for each $\mathbf{k}$ ($k > 1$) it is proven that if $f \notin \mathrm{Str}(J_{\mathbf{k}})$ then the strong partial clone $[f]_s$ contains a family of strong partial subclones of continuum cardinality. Two slightly different constructions are needed depending on whether f is constant (i.e., there exists $x \in \mathbf{k}$ such that $f(\alpha_i) = x$ for all $\alpha_i \in \mathrm{dom}(f)$) or not, and we provide a sketch of the latter construction.

Let f be an n-ary partial operation not in $\mathrm{Str}(J_{\mathbf{k}})$ and not constant. In the sequel we assume that the partial operation f is n-ary and with domain $\alpha^1, \ldots, \alpha^m \in \mathbf{k}^n$, where $\alpha^i := (a^i_1, \ldots, a^i_n)$, and we let A be the $m \times n$ matrix whose rows are $\alpha^1, \ldots, \alpha^m$. Let us first define the following. Let $\mathrm{Col}(A)$ be the set of columns of A, and $\mathbf{v}_f = f(A) \in \mathbf{k}^m$. For $\mathbf{x} := (x_1, \ldots, x_h) \in \mathbf{k}^h$ and $\ell \geq 1$, let $\mathbf{x}^{\times \ell} \in \mathbf{k}^{h\ell}$ be

$$\mathbf{x}^{\times \ell} = (\underbrace{x_1, \ldots, x_1}_{\ell \text{ times}}, \underbrace{x_2, \ldots, x_2}_{\ell \text{ times}}, \ldots, \underbrace{x_h, \ldots, x_h}_{\ell \text{ times}}),$$

and let $[\mathbf{x}] = \{x_1, \ldots, x_h\}$. For a set $X \subseteq \mathbf{k}$ with

$$X = \{x_1 < \cdots < x_{|X|}\}$$

and $a \in X$, let $\mathrm{next}_X(a) \in X$ be defined by

$$\mathrm{next}_X(a) := \begin{cases} x_{i+1} & \text{if } a = x_i \text{ and } i < |X|, \\ x_1 & \text{if } a = x_{|X|}. \end{cases}$$

Furthermore, for $\mathbf{x} = (x_1, \ldots, x_h) \in [\mathbf{v}_f]^h$ and $1 \leq i \leq h$, let $c_i(\mathbf{x})$ be the tuple

$$c_i(\mathbf{x}) := (x_1, \ldots, x_{i-1}, \mathrm{next}_{[\mathbf{v}_f]}(x_i), x_{i+1}, \ldots, x_h).$$

Since the partial operation f is non-constant, the set $[\mathbf{v}_f]$ contains at least two different elements, and so $c_i(\mathbf{x}) \neq \mathbf{x}$ for all $\mathbf{x} \in [\mathbf{v}_f]^h$ and all $i = 1, \ldots, h$. Let $t \geq 0$ be the number of columns $\underset{\sim}{u}$ in the matrix A that satisfy $[\underset{\sim}{u}] = [\mathbf{v}_f]$. Without loss of generality, assume that all those t columns (if any) are the first columns to the left of A.

For each $\ell \geq 1$, define the relation ρ_ℓ of arity ℓd_f by

$$\rho_\ell := \{c_i(\mathbf{v}_f^{\times \ell}) \mid 1 \leq i \leq \ell d_f\} \cup \{\underset{\sim}{u}^{\times \ell} \mid \underset{\sim}{u} \in \mathrm{Col}(A)\}.$$

Notice that $|\rho_\ell| = \ell d_f + n$.

Let M_ℓ be the matrix with ℓd_f rows, whose $(\ell d_f + n)$ columns are the tuples of ρ_ℓ in the following order:

$$c_1(\mathbf{v}_f^{\times \ell}), \ldots, c_{\ell d_f}(\mathbf{v}_f^{\times \ell}), \underset{\sim}{u}_1{}^{\times \ell}, \ldots, \underset{\sim}{u}_n{}^{\times \ell},$$

where $\underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_n$ are the columns of A written in the same order as they appear in A. By $f^{\times \ell}$ we denote the $(\ell d_f + n)$-ary partial operation whose domain is the set of all rows of M_ℓ and defined by

$$f^{\times \ell}(M_\ell) = \mathbf{v}_f^{\times \ell}.$$

Notice that for every $\mathbf{x} = (x_1, \ldots, x_{\ell d_f + n}) \in \mathrm{dom}(f^{\times \ell})$, we have that $x_1, \ldots, x_{\ell d_f + t} \in [\mathbf{v}_f]$.

Example 5. *Let $\mathbf{k} = \{0, 1, 2\}$, $\ell = 3$ and*

$$f \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then $\mathbf{v}_f = (0, 0, 1)$, $\mathbf{v}_f^{\times 3} = (0, 0, 0, 0, 0, 0, 1, 1, 1)$,

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix},$$

$\mathrm{Col}(A) = \{(0, 1, 0)^T, (0, 0, 0)^T, (0, 1, 2)^T\}$, *and* $f^{\times 3}(M_3) =$

$$f^{\times 3} \left(\begin{array}{ccccccccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 2 \end{array}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

This construction yields the following results.

Lemma 11 ([16])**.** *For every $\ell \geq 1$, $f^{\times \ell} \in [f]_s$. Moreover, for $\ell' \geq 1$, $f^{\times \ell} \in \mathrm{pPol}\rho_{\ell'}$ iff $\ell \neq \ell'$.*

Corollary 12. *Let C be a strong partial clone on $\mathbf{k}$ and suppose that C contains a partial operation $f \notin \mathrm{Str}(J_{\mathbf{k}})$ that is not a constant operation. Then the set of strong partial clones contained in C is of continuum cardinality.*

5 Open Questions

The study of fine-grained complexity is still in its infancy and we have only concentrated a handful of results relevant in the context of partial polymorphisms. Now we present a few concrete questions arising from the results presented thus far.

On the non-existence of minimal strong partial clones: We provided a sketch of the construction from Couceiro *et al.* [16], which shows that there for any non-constant $f \notin \mathrm{Str}(J_{\{0,1\}})$ exists $g \notin \mathrm{Str}(J_{\{0,1\}})$ such that $[g]_s \subset [f]_s$. Assuming $\mathsf{T}(\mathrm{Inv}(\{f\})) < 1$, can this construction be used to find g such that $\mathsf{T}(\mathrm{Inv}(\{f\})) < \mathsf{T}(\mathrm{Inv}(\{g\}))$?

Maximal elements of $\mathcal{I}_{\mathrm{Str}}(J_{\mathbf{k}})$: We have seen that $\mathcal{I}_{\mathrm{Str}}(J_{\{0,1\}})$ has a largest element $\mathrm{pPol}(R^{\neq\neq\neq 01}_{1/3})$, resulting in the "easiest NP-complete SAT problem" $\mathrm{SAT}(\{R^{\neq\neq\neq 01}_{1/3}\})$. Given the non-existence of minimal strong partial clones one might be sceptical about the existence of maximal elements of $\mathcal{I}_{\mathrm{Str}}(J_{\{0,1\}})$. However, such elements do in fact exist, one can for example prove that $\mathrm{pPol}(\{R^{\neq\neq\neq 01}_{1/3}, \{(0)\}\})$ and $\mathrm{pPol}(\{R^{\neq\neq\neq 01}_{1/3}, \{(1)\}\})$ are both maximal elements. The caveat here is that $T(\{R^{\neq\neq\neq 01}_{1/3}\}) = \mathsf{T}(\{R^{\neq\neq\neq 01}_{1/3}, \{(0)\}) = \mathsf{T}(\{R^{\neq\neq\neq 01}_{1/3}, \{(1)\}),$ implying that these elements are not interesting from a complexity theoretical point of view. This raises the question of whether there exists a maximal element $\mathrm{pPol}(\Gamma)$ of $\mathcal{I}_{\mathrm{Str}}(J_{\{0,1\}})$ such that $\mathsf{T}(\{R^{\neq\neq\neq 01}_{1/3}\}) < \mathsf{T}(\Gamma)$.

Strong Maltsev conditions and partial polymorphisms: Lagerkvist & Wahlström [30] propose a usage of partial polymorphisms which is similar to how strong Maltsev conditions are used to characterize the classical complexity of CSPs. For example, given the identities defining a Maltsev operation $m(x,x,y) \approx y, m(x,y,y) \approx x$ one for every $\mathbf{k}$ we can define a partial operation f such that $\mathrm{dom}(f) = \{(x,x,y),(x,y,y) \mid x,y \in \mathbf{k}\}$ and such that $f(x,x,y) = y$ and $f(x,y,y) = x$ for all values where it is defined. The operation f is then called a *partial Maltsev operation*. The objective is then, given a partial operation f constructed in this manner, to construct an algorithm for $\mathrm{CSP}(\mathrm{Inv}(f))$ with a running time better than $O(k^n)$. Surprisingly, this is indeed possible for the partial Maltsev operation, where one obtains the upper bound $O(k^{\frac{n}{2}})$. An interesting continuation to this line of research is to consider the identities defining near unanimity operations and edge operations, and investigate if similar improved bounds can be obtained for the corresponding partial operations.

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