The Inclusion Structure of Boolean Weak Bases

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Abstract

Strong partial clones are composition closed sets of partial operations containing all partial projections, characterizable as *partial polymorphisms* of sets of relations Γ ($\mathrm{pPol}(\Gamma)$). If C is a clone it is known that the set of all strong partial clones whose total component equals C, has a greatest element $\mathrm{pPol}(\Gamma_w)$, where Γ_w is called a *weak base*. Weak bases have seen applications in computer science due to their usefulness for proving complexity classifications for constraint satisfaction related problems. In this paper we completely describe the inclusion structure between $\mathrm{pPol}(\Gamma_w)$, $\mathrm{pPol}(\Delta_w)$ for all Boolean weak bases Γ_w and Δ_w.

1 introduction

A *clone* is a set of operations closed under composition which contains all projections. In the last decades clone theory has received quite some attention due to its relevance for classifying the complexity of computational problems such as *constraint satisfaction problems* (CSPs) [1]. This approach is based on the fact that a clone can be described as the set of *polymorphisms* (intuitively, a generalisation of a homomorphism) of a set of relations, which corresponds to a closure operator on relations, closure under *primitive positive definitions*, that can be used to obtain reductions between CSPs. Not all computational problems are compatible with polymorphisms, in the sense that a clone corresponding to a constraint language unequivocally determines whether the problem is tractable, in P, or intractable (typically NP-hard or co-NP-hard). Some examples include the inverse satisfiability problem, enumerating models of CSP with polynomial delay, and the surjective CSP problem [7]. The complexities of several of these problems have been settled, but using non-algebraic proofs based on a large number of case analyses, only valid for the Boolean domain. Schnoor & Schnoor [16] argued that the complexity of such problems is better studied using *partial polymorphisms*, since these correspond to a more restricted closure operator on relations, closure under *quantifier-free primitive positive definitions*. These notions will be formally defined in Section 2 and at the moment we simply view the set of partial polymorphisms of a set of relations Γ, $\mathrm{pPol}(\Gamma)$, as polymorphisms that may be undefined for certain sequences of arguments. Unfortunately, the resulting closed classes of partial operations, *strong partial clones*, are largely unexplored even in the Boolean domain.

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To mitigate this Schnoor & Schnoor introduced the concept of a *weak base* [16] corresponding to a clone C — a relational description of the largest strong partial clone whose total component equals C — and proved that weak bases always exist. Hence, if Γ_w is a weak base corresponding to a clone C then $\mathrm{pPol}(\Gamma_w)$ is the largest set of partial operations not containing a total operation outside of C. The practical motivation behind weak bases is that they offer a considerable simplification for proving hardness results, in the following sense. Assume that $X(\Gamma)$ is a computational problem parameterized by a set of relations Γ, and that we want to determine how Γ influences the complexity of $X(\Gamma)$. Then, instead of proving hardness results for $X(\Gamma)$ for every $\mathrm{pPol}(\Gamma)$ corresponding to a clone C, it is sufficient to show that $X(\Gamma_w)$ is intractable for the weak base Γ_w corresponding to C [16]. The reason is that Γ_w is the least expressive language corresponding to C with respect to quantifier-free primitive positive definitions, and $X(\Gamma_w)$ then intuitively represents the "easiest" problem corresponding to C. For arbitrary finite domains little is known concerning weak bases, but in the Boolean domain they are completely described [11]. Weak bases have successfully been used to prove complexity dichotomies for several different computational problems [2, 3, 12, 16, 17].

In this paper we study additional properties of Boolean weak bases, with a particular focus on their inclusion structure. More precisely, if we let $\mathcal{L}_{\mathcal{W}} = \{\mathrm{pPol}(\Gamma_w) \mid \Gamma_w \text{ is a Boolean weak base}\}$ we are interested in determining the poset $(\mathcal{L}_{\mathcal{W}}, \subseteq)$. Such a classification can be of practical interest since it effectively reduces the number of distinct cases one needs to consider to prove a complexity dichotomy for a computational problem. Determining this inclusion structure is conceptually not difficult, but is in practice rather challenging due to the large number of cases that need to be considered. We propose a method where, given a weak base Γ_w corresponding to a clone C, one effectively needs to consider only the clones *covering* C, i.e., situated directly above in the clone lattice, rather than all clones containing C. Using this method, we in Section 3 completely describe the poset $\mathcal{L}_{\mathcal{W}}$. We are also able to leverage our method to prove a covering result for $\mathrm{pPol}(\Gamma_w)$ and $\mathrm{pPol}(\Delta_w)$ for two specific weak bases Γ_w and Δ_w. This result can be translated to the following statement: the strong partial clone $\mathrm{pPol}(\Gamma_w)$, the set of all partial operations which cannot define a (non-projective) total operation, is covered by the submaximal strong partial clone $\mathrm{pPol}(\Delta_w)$ where $\Delta_w = \{\{(0, 1, 0, 1), (1, 0, 0, 1)\}\}$, but is not covered by any other strong partial clone. Most likely, covering results can also be obtained for other pairs of weak bases, and we discuss this and other open questions in Section 4.

2 Preliminaries

2.1 Partial Operations and Strong Partial Clones

A *k-ary partial operation* over a set D is a map $f\colon \mathrm{dom}(f) \to D$ where $\mathrm{dom}(f) \subseteq D^k$ ($k \geq 1$). We write PAR_D, respectively OP_D, for the set of all partial, respectively total, operations over the set D, and let $\mathsf{BF} = \mathrm{OP}_{\{0,1\}}$. If $f, g \in \mathrm{PAR}_D$, both of arity k, then g is a *suboperation* of f if $\mathrm{dom}(g) \subseteq \mathrm{dom}(f)$ and $g(\mathbf{x}) = f(\mathbf{x})$ for each $\mathbf{x} \in \mathrm{dom}(g)$. Partial operations compose together in a natural way, and if $f, g_1, \ldots, g_m \in \mathrm{PAR}_D$ are partial operations such that f has arity $m \geq 1$ and each g_i arity $n \geq 1$ then we write $f \circ g_1, \ldots, g_m$ for the n-ary partial operation $f(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$ which is defined for $(x_1, \ldots, x_n) \in D^n$ if and only if $(x_1, \ldots, x_n) \in \bigcap_{1 \leq i \leq m} \mathrm{dom}(g_i)$ and $(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n)) \in \mathrm{dom}(f)$. Note that since a total operation can be viewed as a special case of partial operation the above definition is valid also in the total setting. For $k \geq 1$ and $1 \leq i \leq k$ the *ith projection* π_i^k is defined as

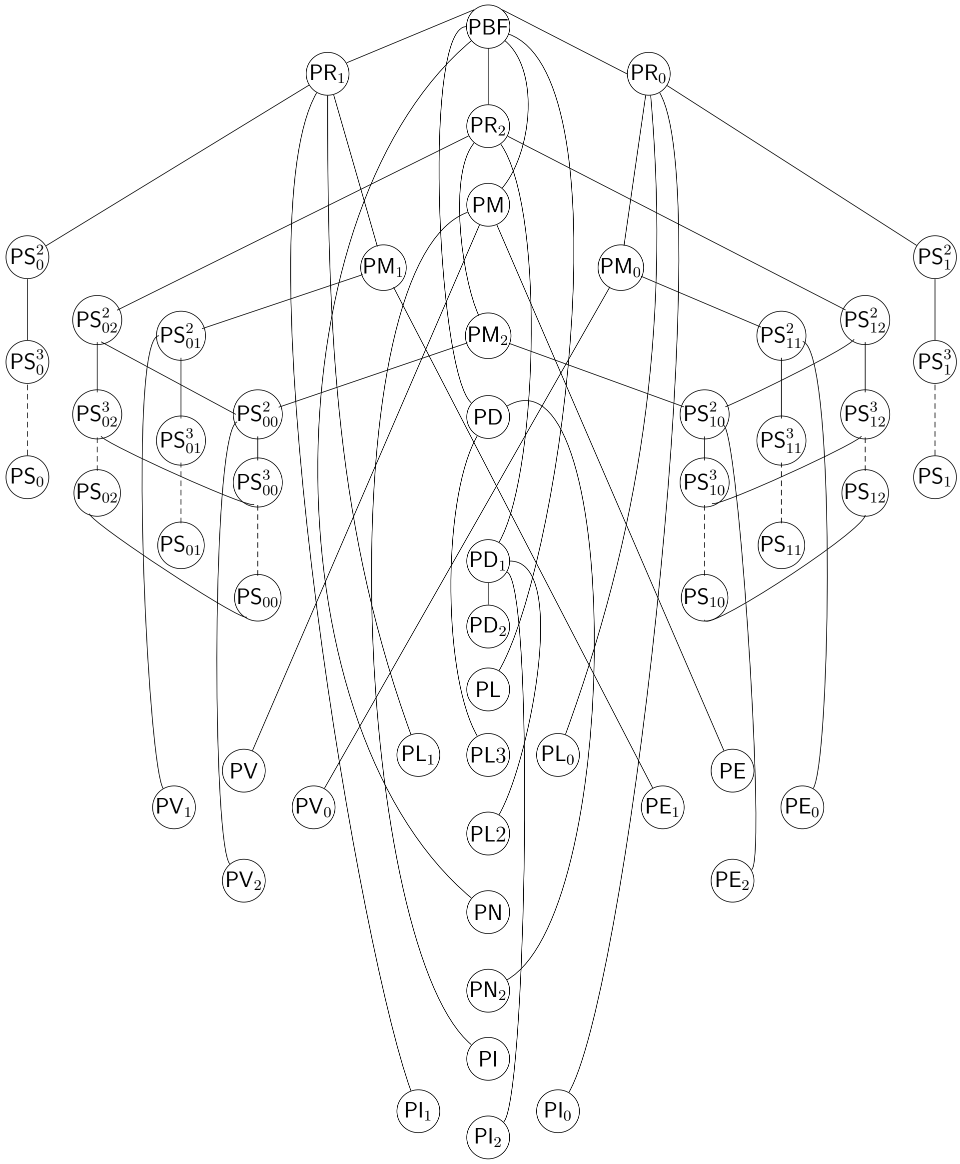

Figure 1: A visualization of the poset $(\mathcal{L}_{\mathcal{W}}, \subseteq)$. A path consisting of upward edges connecting PC to PC' if and only if $\mathsf{PC} \subset \mathsf{PC}'$.

Table 1: Weak bases of Boolean co-clones. The rightmost column contains a base of the corresponding clone.

C	Weak base of IC	Base of C
BF	$\{\mathrm{Eq}_{\{0,1\}}(x_1, x_2)\}$	$\{x_1 \bar{\wedge} x_2\}$
R_0	$\{\mathrm{F}(c_0)\}$	$\{x_1 \wedge x_2, x_1 \oplus x_2\}$
R_1	$\{\mathrm{T}(c_1)\}$	$\{x_1 \vee x_2, x_1 \leftrightarrow x_2\}$
R_2	$\{\mathrm{F}(c_0) \wedge \mathrm{T}(c_1)\}$	$\{x_1 \vee x_2, x_1 \wedge (x_2 \leftrightarrow x_3\}$
M	$\{(x_1 \rightarrow x_2)\}$	$\{x_1 \vee x_2, x_1 \wedge x_2, 0, 1\}$
M_0	$\{(x_1 \rightarrow x_2) \wedge \mathrm{F}(c_0)\}$	$\{x_1 \vee x_2, x_1 \wedge x_2, 0\}$
M_1	$\{(x_1 \rightarrow x_2) \wedge \mathrm{T}(c_1)\}$	$\{x_1 \vee x_2, x_1 \wedge x_2, 1\}$
M_2	$\{(x_1 \rightarrow x_2) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)\}$	$\{x_1 \vee x_2, x_1 \wedge x_2\}$
$\mathsf{S}_0^n, n \geq 2$	$\{\mathrm{OR}^n(x_1, \ldots, x_n) \wedge \mathrm{T}(c_1)\}$	$\{x \rightarrow y, \mathrm{dual}(h_n)\}$
S_0	$\{\mathrm{OR}^n(x_1, \ldots, x_n) \wedge \mathrm{T}(c_1) \mid n \geq 2\}$	$\{x \rightarrow y\}$
$\mathsf{S}_{02}^n, n \geq 2$	$\{\mathrm{OR}^n(x_1, \ldots, x_n) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)\}$	$\{x \vee (y \wedge \neg z), \mathrm{dual}(h_n)\}$
S_{02}	$\{\mathrm{OR}^n(x_1, \ldots, x_n) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1) \mid n \geq 2\}$	$\{x \vee (y \wedge \neg z)\}$
$\mathsf{S}_{01}^n, n \geq 2$	$\{\mathrm{OR}^n(x_2, \ldots, x_{n+1}) \wedge (x_1 \rightarrow x_2 \cdots x_{n+1}) \wedge \mathrm{T}(c_1)\}$	$\{\mathrm{dual}(h_n), 1\}$
S_{01}	$\{\mathrm{OR}^n(x_2, \ldots, x_{n+1}) \wedge (x_1 \rightarrow x_2 \cdots x_{n+1}) \wedge \mathrm{T}(c_1) \mid n \geq 2\}$	$\{x \vee (y \wedge z), 1\}$
$\mathsf{S}_{00}^n, n \geq 2$	$\{\mathrm{OR}^n(x_2, \ldots, x_{n+1}) \wedge (x_1 \rightarrow x_2 \cdots x_{n+1}) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)\}$	$\{x \vee (y \wedge z), \mathrm{dual}(h_n)\}$
S_{00}	$\{\mathrm{OR}^n(x_1, \ldots, x_n) \wedge (x \rightarrow x_1 \cdots x_n) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1) \mid n \geq 2\}$	$\{x \vee (y \wedge z)\}$
$\mathsf{S}_1^n, n \geq 2$	$\{\mathrm{NAND}^n(x_1, \ldots, x_n) \wedge \mathrm{F}(c_0)\}$	$\{x \wedge \neg y, h_n\}$
S_1	$\{\mathrm{NAND}^n(x_1, \ldots, x_n) \wedge \mathrm{F}(c_0) \mid n \geq 2\}$	$\{x \wedge \neg y\}$
$\mathsf{S}_{12}^n, n \geq 2$	$\{\mathrm{NAND}^n(x_1, \ldots, x_n) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)\}$	$\{x \wedge (y \vee \neg z), h_n\}$
S_{12}	$\{\mathrm{NAND}^n(x_1, \ldots, x_n) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1) \mid n \geq 2\}$	$\{x \wedge (y \vee \neg z)\}$
$\mathsf{S}_{11}^n, n \geq 2$	$\{\mathrm{NAND}^n(x_2, \ldots, x_{n+1}) \wedge (x_2 \rightarrow x_1) \wedge \ldots \wedge (x_{n+1} \rightarrow x_1) \wedge \mathrm{F}(c_0)\}$	$\{h_n, 0\}$
S_{11}	$\{\mathrm{NAND}^n(x_2, \ldots, x_{n+1}) \wedge (x_2 \rightarrow x_1) \wedge \ldots \wedge (x_{n+1} \rightarrow x_1) \wedge \mathrm{F}(c_0) \mid n \geq 2\}$	$\{x \wedge (y \vee z), 0\}$
$\mathsf{S}_{10}^n, n \geq 2$	$\{\mathrm{NAND}^n(x_2, \ldots, x_{n+1}) \wedge (x_2 \rightarrow x_1) \wedge \ldots \wedge (x_{n+1} \rightarrow x_1) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)\}$	$\{x \wedge (y \vee z), h_n\}$
S_{10}	$\{\mathrm{NAND}^n(x_2, \ldots, x_{n+1}) \wedge (x_2 \rightarrow x_1) \wedge \ldots \wedge (x_{n+1} \rightarrow x_1) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1) \mid n \geq 2\}$	$\{x \wedge (y \vee z)\}$
D	$\{\mathrm{Neq}(x_1, x_2)\}$	$\{x_1\overline{x_2} \vee x_1\overline{x_3} \vee \overline{x_2}\,\overline{x_3}\}$
D_1	$\{\mathrm{Neq}(x_1, x_2) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)\}$	$\{x_1 x_2 \vee x_1\overline{x_3} \vee x_2\overline{x_3}\}$
D_2	$\{\mathrm{OR}^2(x_2, x_4) \wedge \mathrm{Neq}(x_2, x_3) \wedge \mathrm{Neq}(x_4, x_1) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)\}$	$\{h_2\}$
L	$\{\mathrm{EV}^4(x_1, x_2, x_3, x_4)\}$	$\{x_1 \oplus x_2, 1\}$
L_0	$\{\mathrm{EV}^3(x_1, x_2, x_3) \wedge \mathrm{F}(c_0)\}$	$\{x_1 \oplus x_2\}$
L_1	$\{\mathrm{OD}^3(x_1, x_2, x_3) \wedge \mathrm{T}(c_1)\}$	$\{x_1 \leftrightarrow x_2\}$
L_2	$\{\mathrm{EV}^3_{3\neq}(x_1, \ldots, x_6) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)\}$	$\{x_1 \oplus x_2 \oplus x_3\}$
L_3	$\{\mathrm{EV}^4_{4\neq}(x_1, \ldots, x_8)\}$	$\{x_1 \oplus x_2 \oplus x_3 \oplus 1\}$
V	$\{(\overline{x_4} \leftrightarrow \overline{x_2}\,\overline{x_3}) \wedge (\overline{x_2} \vee \overline{x_3} \rightarrow \overline{x_1})\}$	$\{x_1 \vee x_2, 0, 1\}$
V_0	$\{(\overline{x_1} \leftrightarrow \overline{x_2}\,\overline{x_3}) \wedge \mathrm{F}(c_0)\}$	$\{x_1 \vee x_2, 0\}$
V_1	$\{(\overline{x_4} \leftrightarrow \overline{x_2}\,\overline{x_3}) \wedge (\overline{x_2} \vee \overline{x_3} \rightarrow \overline{x_1}) \wedge \mathrm{T}(c_1)\}$	$\{x_1 \vee x_2, 1\}$
V_2	$\{(\overline{x_1} \leftrightarrow \overline{x_2}\,\overline{x_3}) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)\}$	$\{x_1 \vee x_2\}$
E	$\{(x_1 \leftrightarrow x_2 x_3) \wedge (x_2 \vee x_3 \rightarrow x_4)\}$	$\{x_1 \wedge x_2, 0, 1\}$
E_0	$\{(x_1 \leftrightarrow x_2 x_3) \wedge (x_2 \vee x_3 \rightarrow x_4) \wedge \mathrm{F}(c_0)\}$	$\{x_1 \wedge x_2, 0\}$
E_1	$\{(x_1 \leftrightarrow x_2 x_3) \wedge \mathrm{T}(c_1)\}$	$\{x_1 \wedge x_2, 1\}$
E_2	$\{(x_1 \leftrightarrow x_2 x_3) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)\}$	$\{x_1 \wedge x_2\}$
N	$\{\mathrm{EV}^4(x_1, x_2, x_3, x_4) \wedge x_1 x_4 \leftrightarrow x_2 x_3\}$	$\{\overline{x_1}, 0, 1\}$
N_2	$\{\mathrm{EV}^4_{4\neq}(x_1, \ldots, x_8) \wedge x_1 x_4 \leftrightarrow x_2 x_3\}$	$\{\overline{x_1}\}$
I	$\{(x_1 \leftrightarrow x_2 x_3) \wedge (\overline{x_4} \leftrightarrow \overline{x_2}\,\overline{x_3})\}$	$\{\pi_1^1, 0, 1\}$
I_0	$\{(\overline{x_1} \vee \overline{x_2}) \wedge (\overline{x_1}\,\overline{x_2} \leftrightarrow \overline{x_3}) \wedge \mathrm{F}(c_0)\}$	$\{\pi_1^1, 0\}$
I_1	$\{(x_1 \vee x_2) \wedge (x_1 x_2 \leftrightarrow x_3) \wedge \mathrm{T}(c_1)\}$	$\{\pi_1^1, 1\}$
I_2	$\{R^{1/3}_{3\neq}(x_1, \ldots, x_6) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)\}$	$\{\pi_1^1\}$

$\pi_i^k(x_1, \ldots, x_i, \ldots, x_k) = x_i$, and a suboperation of a projection is a a *partial projection.*

Definition 1. *A set* $\mathsf{C} \subseteq \mathrm{OP}_D$ *is a* clone *if* C *contains all projections over* D *and* C *is closed under composition, and a set* $\mathsf{P} \subseteq \mathrm{PAR}_D$ *is a* strong partial clone *if* P *contains all partial projections over* D *and* P *is closed under composition.*

If F is a set of (partial) operations then we write $[F]$ (respectively $[F]_s$) for the intersection of all (strong partial) clones containing F, and say that F is a *base*.

2.2 Partial Polymorphisms and Relations

Clones and strong partial clones can also be described through relations. First, let Rel_D be the set of all (finitary) relations over $D \subseteq \mathbb{N}$. Then, given a k-ary relation $R \in \mathrm{Rel}_D$ and an n-ary partial operation $f \in \mathrm{PAR}_D$ we say that f *preserves* R, or that R is *invariant* under f, if for each sequence $t_1, \ldots, t_n \in R$ it holds that either $f(t_1, \ldots, t_n) := (f(t_1[1], \ldots, t_n[1]), \ldots, f(t_1[k], \ldots, t_n[k])) \in R$ or that there exists i such that $(t_1[i], \ldots, t_n[i]) \notin \mathrm{dom}(f)$ (where $t_i[j]$ is the jth element of t_i).

If we then let $\mathrm{Pol}(\Gamma)$ (respectively $\mathrm{pPol}(\Gamma)$) be the set of all (partial) operations preserving each relation in Γ, it is easy to verify that $\mathrm{pPol}(\Gamma)$ forms a strong partial clone and that $\mathrm{Pol}(\Gamma)$

Table 2: Relations.

Relation	Definition
F	$\{(0)\}$
T	$\{(1)\}$
Neq	$\{(0,1),(1,0)\}$
EV^n	$\{(x_1,\ldots,x_n) \in \{0,1\}^n \mid x_1+\ldots+x_n \text{ is even}\}$
$\mathrm{EV}^n_{n\neq}$	$\mathrm{EV}^n(x_1,\ldots,x_n) \wedge \mathrm{Neq}(x_1,x_{n+1}) \wedge \ldots \wedge \mathrm{Neq}(x_n,x_{2n})$
OD^n	$\{(x_1,\ldots,x_n) \in \{0,1\}^n \mid x_1+\ldots+x_n \text{ is odd}\}$
$\mathrm{OD}^n_{n\neq}$	$\mathrm{OD}^n(x_1,\ldots,x_n) \wedge \mathrm{Neq}(x_1,x_{n+1}) \wedge \ldots \wedge \mathrm{Neq}(x_n,x_{2n})$
OR^n	$\{0,1\}^n \setminus \{(0,\ldots,0)\}$
NAND^n	$\{0,1\}^n \setminus \{(1,\ldots,1)\}$
$R^{1/3}_{3\neq}$	$\{(0,0,1,1,1,0),(0,1,0,1,0,1),(1,0,0,0,1,1)\}$

forms a clone. Dually, if $F \subseteq \mathrm{PAR}_D$, we let $\mathrm{Inv}(F) \subseteq \mathrm{Rel}_D$ (sometimes written $\mathsf{I}F$) be the set of all relations invariant under each (partial) operation in F. The operator $\mathrm{Inv}(\cdot)$ relate to $\mathrm{pPol}(\cdot)$ and $\mathrm{Pol}(\cdot)$ in the following sense.

Theorem 2 ([4, 5, 8, 15])**.** *Let Γ and Δ be two sets of relations over a finite set. Then (1) $\Gamma \subseteq \mathrm{Inv}(\mathrm{Pol}(\Delta))$ if and only if $\mathrm{Pol}(\Delta) \subseteq \mathrm{Pol}(\Gamma)$, and (2) $\Gamma \subseteq \mathrm{Inv}(\mathrm{pPol}(\Delta))$ if and only if $\mathrm{pPol}(\Delta) \subseteq \mathrm{pPol}(\Gamma)$.*

It is sometimes easier to work with $\mathrm{Inv}(F)$ directly instead of invoking its corresponding (strong partial) clone. Fortunately, these are well-behaved sets of relations, in the following sense: if F consists of total operations, then $\mathrm{Inv}(F)$ is closed under formation of first-order formulas consisting of existential quantification, conjunction, and equality constraints, *primitive positive definitions* (pp-definitions). To make this a bit more precise, first observe that the set of models of a first-order formula $\varphi(x_1,\ldots,x_n)$ can be viewed as a relation R, and we then write $R(x_1,\ldots,x_n) \equiv \varphi(x_1,\ldots,x_n)$ for $R = \{(f(x_1),\ldots,f(x_n)) \mid f \text{ is a model of } \varphi(x_1,\ldots,x_n)\}$. Then, if $\Gamma \subseteq \mathrm{Rel}_D$, a primitive positive definition of an n-ary $R \in \mathrm{Rel}_D$ over Γ is simply the condition that $R(x_1,\ldots,x_n) \equiv \exists y_1,\ldots,y_{n'} \colon R_1(\mathbf{x}_1) \wedge \ldots \wedge R_m(\mathbf{x}_m)$ where each $R_i \in \Gamma \cup \{\mathrm{Eq}_D\}$ and each $\mathbf{x}_i$ is a tuple of variables over $x_1,\ldots,x_n,y_1,\ldots,y_{n'}$. Here, $\mathrm{Eq}_D = \{(x,x) \mid x \in D\}$ is the equality relation over D. Similarly, if $F \subseteq \mathrm{PAR}_D$ it is known that $\mathrm{Inv}(F)$ is closed under *quantifier-free primitive positive definitions* (qfpp-definitions) which are simply primitive positive definitions without existential quantification.

If we let $\langle\Gamma\rangle$ (respectively $\langle\Gamma\rangle_{\nexists}$) be the smallest set of relations containing Γ and which is closed under pp-definitions (respectively, qfpp-definitions), then it is known that $\langle\Gamma\rangle = \mathrm{Inv}(\mathrm{Pol}(\Gamma))$ and that $\langle\Gamma\rangle_{\nexists} = \mathrm{Inv}(\mathrm{pPol}(\Gamma))$. The sets $\langle\Gamma\rangle$ and $\langle\Gamma\rangle_{\nexists}$ are furthermore known as *relational clones*, or *co-clones*, and *weak systems*, or *weak co-clones*. In both cases we refer to the set Γ as a *base* of $\langle\Gamma\rangle$ or $\langle\Gamma\rangle_{\nexists}$. Our main usage of this correspondence in this paper will be to show an inclusion of the form $\mathrm{pPol}(\Gamma) \subseteq \mathrm{pPol}(\Delta)$ by proving that each relation in Δ is qfpp-definable over Γ.

2.3 Intervals of Strong Partial Clones and Weak Bases

The lattice of strong partial clones is of continuum cardinality even in the Boolean domain. The maximal elements have been determined [13][Section 20.4] and recently it was also proven that no minimal elements can exist [6], but in more general terms a complete understanding is still out of reach. A slightly more manageable strategy is to first fix a clone C and then describe the set of all strong partial clones corresponding to the clone C, motivating the following definition.

Definition 3. *Let C be a clone over a set D. We define the set $\mathcal{L}_{D|\mathsf{C}} = \{\text{pPol}(\Gamma) \mid \Gamma \subseteq \text{Rel}_D, \text{Pol}(\Gamma) = \mathsf{C}\}$.*

Hence, $\mathcal{L}_{D|\mathsf{C}}$ is the set of all strong partial clones over D whose total component equals the given clone C. Say that a clone C is *finitely related* if there exists a finite $\Gamma \subseteq \text{Rel}_D$ such that $\text{Pol}(\Gamma) = \mathsf{C}$. Schnoor & Schnoor [16] proved that if C is finitely related then $\mathcal{L}_{D|\mathsf{C}}$ has a greatest element, namely the union of all members of $\mathcal{L}_{D|\mathsf{C}}$.

Theorem 4. *[16] Let C be a clone over a finite set D. If C is finitely related, then $(\bigcup_{\mathsf{P} \in \mathcal{L}_{D|\mathsf{C}}}^{\infty} \mathsf{P}) \cap \text{OP}_D = \mathsf{C}$.*

The fact that a greatest element exists motivates the following definition.

Definition 5. *Let C be a clone over D. We say that $\Gamma \subseteq \text{Rel}_D$ is a* weak base *of* $\text{Inv}(\mathsf{C})$ *if* $\text{pPol}(\Gamma) = (\bigcup_{\mathsf{P} \in \mathcal{L}_{D|\mathsf{C}}}^{\infty} \mathsf{P})$.

In relational terms Definition 5 then implies that $\langle \Gamma \rangle_{\not\exists} \subseteq \langle \Delta \rangle_{\not\exists}$ for each base Δ of $\text{Inv}(\mathsf{C})$. Hence, a weak base is a base of $\text{Inv}(\mathsf{C})$ minimally expressive with respect to qfpp-definitions. Boolean weak bases were fully described by Lagerkvist [11] and we refer the reader to Table 2.1 for a comprehensive list. Each entry consists of a Boolean clone C, a weak base of IC, and a base of C. Here, and in the sequel, we will define Boolean relations and operations by logical formulas and employ infix notation whenever convenient. Variables are typically named $x_1, \ldots, x_n$ or x, y, z, with the exception of variables which are assigned constant values 0 and 1. These are instead denoted by c_0 and c_1, respectively, and we typically assume that c_0 occurs as the first argument and c_1 as the last. For example, we write $\overline{x}$ for $f(0) = 1, f(1) = 0$, $x_1 \bar{\wedge} x_2$ for $f(0,0) = 1$, $f(0,1) = f(1,0) = f(1,1) = 0$, and $x_1 \leftrightarrow x_2$ for $f(0,0) = 1$, $f(0,1) = f(1,0) = 0$, $f(1,1) = 1$. In addition, we will frequently write $x_1 \cdots x_n$ instead of $x_1 \wedge \ldots \wedge x_n$, and write 0 and 1 for the two constant Boolean operations. Hence, logical formulas are used to denote both operations and relations, but the intended meaning will always be clear from the context. For example, the entry for the clone V in Table 2.1 consists of the base $\{x_1 \vee x_2, 0, 1\}$ and the logical formula $(\overline{x_4} \leftrightarrow \overline{x_2}\,\overline{x_3}) \wedge (\overline{x_2} \vee \overline{x_3} \rightarrow \overline{x_1})$ which defines the 4-ary relation $\{(0,0,0,0), (1,0,1,0), (1,1,0,0), (1,1,1,0), (1,1,1,1)\}$ which is a weak base of IV.

In addition, for each $n \geq 2$, we let $h_n(x_1, \ldots, x_{n+1}) = \bigvee_{i=1}^{n+1} x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1}$, and for each n-ary Boolean operation f, we let $\text{dual}(f)(x_1, \ldots, x_n) = \overline{f(\overline{x_1}, \ldots, \overline{x_n})}$. Definitions of the additional relations used in Table 2.1 can be found in Table 2.

3 Structure of Boolean Weak Bases

Given a Boolean weak base Γ_w, our goal is to describe every weak base Δ_w such that $\text{pPol}(\Gamma_w) \subset \text{pPol}(\Delta_w)$. To simplify the notation, given a Boolean clone C, we write WC for the weak base of $\text{Inv}(\mathsf{C})$ from Table 2.1 and PC for $\text{pPol}(\mathsf{WC})$. Furthermore, let $\mathcal{L}_W = \{\mathsf{PC} \mid \mathsf{C}$ is a Boolean clone$\}$. Hence, we need to describe the set $\mathcal{L}_W$ with respect to the partial order $\subseteq$. At a first glance this problem might appear to be straightforward due to Table 2.1 in combination with Post's lattice of Boolean clones [14]. What one needs to do is to, for every Boolean clone C and every Boolean clone C' such that $\mathsf{C} \subset \mathsf{C}'$, verify whether the inclusion $\mathsf{PC} \subset \mathsf{PC}'$ holds or not. This can

Table 3: Qfpp-definition of WC_2 over WC_1.

C_2	C_1	$\mathsf{WC}_2 \in \langle\{\mathsf{WC}_1\}\rangle_{\nexists}$
D_1	I_2	$\mathsf{WD}_1(c_0,x_1,x_2,c_1) \equiv \mathsf{WI}_2(c_0,c_0,x_1,x_2,c_1,x_2,c_1,c_1)$
R_0	I_0	$\mathsf{WR}_0(c_0) \equiv \mathsf{WI}_0(c_0,c_0,c_0,c_0)$
R_1	I_1	$\mathsf{WR}_1(c_1) \equiv \mathsf{WI}_1(c_1,c_1,c_1,c_1)$
M	I	$\mathsf{WM}(x_1,x_2) \equiv \mathsf{WI}(x_1,x_1,x_2,x_2)$
D	N_2	$\mathsf{WD}(x_1,x_2) \equiv \mathsf{WN}_2(x_1,x_2,x_1,x_2,x_1,x_2,x_1,x_2)$
S_{00}^2	V_2	$\mathsf{WS}_{00}^2(c_0,x_1,x_2,x_3,c_1) \equiv \mathsf{WV}_2(c_0,x_2,x_3,c_1,c_1) \wedge \mathsf{WV}_2(c_0,x_2,x_1,x_2,c_1) \wedge \mathsf{WV}_2(c_0,x_3,x_1,x_3,c_1)$
M_0	V_0	$\mathsf{WM}_0(c_0,x_1,x_2) \equiv \mathsf{WV}_0(c_0,x_1,x_2,x_2)$
S_{01}^2	V_1	$\mathsf{WS}_{01}^2(x_1,x_2,x_3,c_1) \equiv \mathsf{WV}_1(x_1,x_2,x_3,c_1,c_1)$
M	V	$\mathsf{WM}(x_1,x_2) \equiv \mathsf{WV}(x_2,x_2,x_1,x_1)$
S_{10}^2	E_2	$\mathsf{WS}_{10}^2(c_0,x_1,x_2,x_3,c_1) \equiv \mathsf{WE}_2(c_0,c_0,x_2,x_3,c_1) \wedge \mathsf{WE}_2(c_0,x_2,x_1,x_2,c_1) \wedge \mathsf{WE}_2(c_0,x_3,x_1,x_3,c_1)$
S_{11}^2	E_0	$\mathsf{WS}_{11}^2(c_0,x_1,x_2,x_3) \equiv \mathsf{WE}_0(c_0,c_0,x_3,x_2,x_1)$
M_1	E_1	$\mathsf{WM}_1(x_1,x_2,c_1) \equiv \mathsf{WE}_1(x_1,x_2,x_2,c_1)$
D_1	L_2	$\mathsf{WD}_1(c_0,x_1,x_2,c_1) \equiv \mathsf{WL}_2(c_0,c_0,x_1,x_1,x_2,x_2,c_1,c_1)$
D	L_3	$\mathsf{WD}(x_1,x_2) \equiv \mathsf{WL}_3(x_1,x_2,x_2,x_1,x_1,x_2,x_1,x_1,x_2)$
R_1	L_1	$\mathsf{WR}_1(c_1) \equiv \mathsf{WL}_1(c_1,c_1,c_1,c_1)$
R_0	L_0	$\mathsf{WR}_0(c_1) \equiv \mathsf{WL}_1(c_0,c_0,c_0,c_0)$
D_1	D_2	$\mathsf{WD}_1(c_0,x_1,x_2,c_1) \equiv \mathsf{WD}_2(c_0,c_1,x_1,c_0,x_2,c_1)$
R_2	D_1	$\mathsf{WR}_2(c_0,c_1) \equiv \mathsf{WD}_1(c_0,c_0,c_1,c_1)$
R_2	M_2	$\mathsf{WR}_2(c_0,c_1) \equiv \mathsf{WM}_2(c_0,c_0,c_0,c_1)$
R_1	M_1	$\mathsf{WR}_1(c_1) \equiv \mathsf{WM}_1(c_1,c_1,c_1)$
R_0	M_0	$\mathsf{WR}_0(c_0) \equiv \mathsf{WM}_0(c_0,c_0,c_0)$
M_2	S_{00}^2	$\mathsf{WM}_2(c_0,x_1,x_2,c_1) \equiv \mathsf{WS}_{00}^2(c_0,x_1,x_2,c_1,c_1)$
M_1	S_{01}^2	$\mathsf{WM}_1(x_1,x_2,c_1) \equiv \mathsf{WS}_{01}^2(x_1,x_2,c_1,c_1)$
R_2	S_{02}^2	$\mathsf{WR}_2(c_0,c_2) \equiv \mathsf{WS}_{02}^2(c_0,c_1,c_1,c_0)$
R_1	S_{0}^2	$\mathsf{WR}_1(c_1) \equiv \mathsf{WS}_{0}^2(c_1,c_1,c_1)$
M_2	S_{10}^2	$\mathsf{WM}_2(c_0,x_1,x_2,c_1) \equiv \mathsf{WS}_{10}^2(c_0,x_1,c_0,x_2,c_1)$
M_0	S_{11}^2	$\mathsf{WM}_0(c_0,x_1,x_2) \equiv \mathsf{WS}_{11}^2(c_0,x_1,c_0,x_2)$
R_2	S_{12}^2	$\mathsf{WR}_2(c_0,c_2) \equiv \mathsf{WS}_{12}^2(c_0,c_0,c_0,c_1)$
R_0	S_{1}^2	$\mathsf{WR}_2(c_0) \equiv \mathsf{WS}_{1}^2(c_0,c_0,c_0)$

be done by either showing that $\mathsf{WC}' \subset \langle \mathsf{WC} \rangle_{\nexists}$, implying that $\mathrm{PC} \subset \mathrm{PC}'$, or by finding a partial operation f preserving WC but not WC'. All inclusions of the former kind are visualised in Figure 1 and are proved in Table 3 and Lemma 8. This figure makes it clear that a large number of inclusions valid in Post's lattice are no longer valid in $\mathcal{L}_{\mathcal{W}}$. However, to prove this rigorously we for each pair of clones C, C' where $\mathsf{C} \subset \mathsf{C}'$ but not connected by an edge in Figure 1, would need to provide an operation preserving WC but not WC'. This is conceptually easy but rather impractical due to the large number of cases that needs to be considered, and we propose a simpler method. First, if $\mathsf{C}_1 \subset \mathsf{C}_2$ are two clones, then C_2 is said to *cover* C_1 if there does not exist a clone C' such that $\mathsf{C}_1 \subset \mathsf{C}' \subset \mathsf{C}_2$, and we let $Cov(\mathsf{C})$ be the set of all clones covering C. We then make the following observation.

Theorem 6. *Let $\mathsf{C}_1 \subset \mathsf{C}_3 \subseteq \mathsf{C}_2$ be Boolean clones such that $\mathsf{C}_3 \in Cov(\mathsf{C}_1)$. If $[\mathrm{PC}_1 \cup \mathsf{C}_3]_s \cap \mathsf{BF} \not\subseteq \mathsf{C}_2$, then $\mathrm{PC}_1 \not\subseteq \mathrm{PC}_2$.*

Proof. If $\mathrm{PC}_1 \subset \mathrm{PC}_2$, then $\mathrm{PC}_2 \supseteq [\mathrm{PC}_1 \cup \mathsf{C}_3]_s$ since $\mathsf{C}_3 \subseteq \mathsf{C}_2$. But then WC_2 cannot be a weak base of $\mathrm{Inv}(\mathsf{C}_2)$ since $\mathrm{Pol}(\mathsf{WC}_2) \neq \mathsf{C}_2$ by the assumption that C_2 does not contain $[\mathrm{PC}_1 \cup \mathsf{C}_3]_s \cap \mathsf{BF}$. Hence, $\mathrm{PC}_1 \not\subseteq \mathrm{PC}_2$. □

The advantage of Theorem 6 is therefore that we in practice only need to consider $Cov(\mathsf{C})$ instead of an arbitrary clone, in order to rule out possible inclusions in $\mathcal{L}_W$. Hence, for each Boolean clone C and $\mathsf{C}' \in Cov(\mathsf{C})$ we need to determine the strong partial clone $[\mathsf{PC} \cup \mathsf{C}']_s$. In other words we need to determine which total operations that are definable using partial polymorphisms of WC together with the new total operations from C'. To this aid we begin by defining the following.

Definition 7. *Let $f, f_1, \ldots, f_m \in \mathrm{OP}_{\{0,1\}}$ be operations of arity n. Define the $(m+n)$-ary partial operation $g^f_{f_1,\ldots,f_m}$ with domain $\mathrm{dom}(g^f_{f_1,\ldots,f_m}) = \{(f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}), \mathbf{x}) \mid \mathbf{x} \in \{0,1\}^n\}$, such that $g^f_{f_1,\ldots,f_m}(f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}), \mathbf{x}) = f(\mathbf{x})$ for each $\mathbf{x} \in \{0,1\}^n$.*

The point of Definition 7 is therefore to construct a partial operation $g^f_{f_1,\ldots,f_m}$ using given operations $f_1, \ldots, f_m$ such that $f \in [\{f_1, \ldots, f_m, g^f_{f_1,\ldots,f_m}\}]_s$. In the case when some f_i does not depend on all its arguments, i.e., there exists $g_i \in [\{f_i\}]$ of arity less than n such that $f_i \in [\{g_i\}]$, we will typically write $g^f_{f_1,\ldots,g_i,\ldots,f_m}$ instead of $g^f_{f_1,\ldots,f_i,\ldots,f_m}$ since the intended ordering of arguments will always be clear from the context. Let us illustrate how this construction can be used together with Theorem 6 by an example.

Example 1. *Consider the three clones $\mathsf{BF}, \mathsf{I}_2, \mathsf{N}_2$. Using the bases from Table 2.1 we have that $\mathsf{BF} = [\{x \bar{\wedge} y\}]$, $\mathsf{I}_2 = [\{\pi^1_1\}]$, and $\mathsf{N}_2 = [\{\overline{x}\}]$. In order to apply Theorem 6 we show that $x \bar{\wedge} y \in [\mathrm{P}\mathsf{I}_2 \cup \{\overline{x}\}]_s$. Let $f(x,y) = x \bar{\wedge} y$ and $f_1(x) = \overline{x}$. Using Definition 7 we construct the ternary partial operation $g^f_{f_1}$, resulting in a partial operation with domain $\{(f_1(x), x, y) \mid x, y \in \{0,1\}\}$ defined such that $g^f_{f_1}(f_1(x), x, y) = x \bar{\wedge} y$ for all $x, y \in \{0,1\}$. In other words $g^f_{f_1}(1,0,0) = 1$ and $g^f_{f_1}(1,0,1) = g^f_{f_1}(0,1,0) = g^f_{f_1}(0,1,1) = 0$, and it is readily verified that $g^f_{f_1}$ preserves WI_2. Theorem 6 then implies that $\mathrm{P}\mathsf{I}_2 \not\subseteq \mathrm{P}\mathsf{C}$ for every clone C such that $\mathsf{N}_2 \subseteq \mathsf{C}$ and $\mathsf{C} \neq \mathsf{BF}$.*

The main technical difficulty is to choose the operations $f_1, \ldots, f_m$ in a suitable way such that the resulting partial operation $g^f_{f_1,\ldots,f_m}$ actually preserves WC. We have organised these definitions in Table 3, which should be interpreted as follows. First, each entry begins with three distinct clones $\mathsf{C}_1, \mathsf{C}_2, \mathsf{C}_3$ where $\mathsf{C}_3 \in \mathcal{C}ov(\mathsf{C}_1)$ and $\mathrm{P}\mathsf{C}_1 \subset \mathrm{P}\mathsf{C}_2$. This is followed by one, or possibly two, operations f, f' such that $[\mathsf{C}_3 \cup \{f, f'\}] = \mathsf{C}_2$. The last element of the entry then consists of operations $f_1, \ldots, f_m, f'_1, \ldots, f'_m \in \mathsf{C}_3$ such that $g^f_{f_1,\ldots,f_m}$ and $g^{f'}_{f'_1,\ldots,f'_m}$ preserve WC_1[1]. Hence, Theorem 6 implies that $\mathrm{P}\mathsf{C}_1 \not\subseteq \mathrm{P}\mathsf{C}'$ for any C' such that $\mathsf{C}_3 \subseteq \mathsf{C}' \subset \mathsf{C}_2$.

Example 2. *Consider the entry in Table 3 for $\mathsf{R}_2, \mathsf{I}_2, \mathsf{E}_2$. Then $f(x,y) = x \vee y$, $f'(x,y,z) = x \wedge (y \leftrightarrow z)$, and $\mathsf{E}_2 = [\{\wedge\}]$. The provided definitions of f_1, f'_1, and f'_2 are $f_1(x,y) = x \wedge y$, $f'_1(x,y,z) = x \wedge y$, and $f'_2(x,y,z) = x \wedge z$, resulting in partial operations $g^f_{f_1}$ and $g^{f'}_{f'_1,f'_2}$ defined such that $g^f_{f_1}(x \wedge y, x, y) = f(x,y) = x \vee y$ and $g^{f'}_{f'_1,f'_2}(f'_1(x,y,z), f'_2(x,y,z), x, y, z) = g^{f'}_{f'_1,f'_2}(x \wedge y, x \wedge z, x, y, z) = f'(x,y,z) = x \wedge (y \leftrightarrow z)$. Hence, $[\mathrm{P}\mathsf{I}_2 \cup \mathsf{E}_2]_s$ contains R_2, and Theorem 6 then implies that $\mathrm{P}\mathsf{I}_2 \not\subseteq \mathrm{P}\mathsf{C}$ for every $\mathsf{E}_2 \subseteq \mathsf{C} \subset \mathsf{R}_2$.*

We now turn to the infinite chains in Post's lattice, i.e., clones C containing S_{00} but contained in S^2_0, or their dual clones S_{10} and S^2_1.

Lemma 8. *Let $n \geq 2$. Then $\mathrm{P}\mathsf{S}^{n+1}_0 \subset \mathrm{P}\mathsf{S}^n_0$, $\mathrm{P}\mathsf{S}^{n+1}_{02} \subset \mathrm{P}\mathsf{S}^n_{02}$, $\mathrm{P}\mathsf{S}^{n+1}_{01} \subset \mathrm{P}\mathsf{S}^n_{01}$, $\mathrm{P}\mathsf{S}^{n+1}_{00} \subset \mathrm{P}\mathsf{S}^n_{00}$, and $\mathrm{P}\mathsf{S}^n_{00} \subset \mathrm{P}\mathsf{S}^n_{02}$. Moreover, $\mathrm{P}\mathsf{C} \not\subseteq \mathrm{P}\mathsf{C}'$ for any other two clones $\mathsf{C}, \mathsf{C}' \in \{\mathsf{S}^n_0, \mathsf{S}^n_{02}, \mathsf{S}^n_{01}, \mathsf{S}^n_{00} \mid n \geq 2\}$.*

Proof. The inclusions can be proved via the qfpp-definitions:

$$\mathsf{WS}^n_0(x_1, \ldots, x_n, c_1) \equiv \mathsf{WS}^{n+1}_0(x_1, x_1, x_2, \ldots, x_n, c_1),$$

$$\mathsf{WS}^n_{02}(c_0, x_1, \ldots, x_n, c_1) \equiv \mathsf{WS}^{n+1}_{02}(c_0, x_1, x_1, x_2, \ldots, x_n, c_1),$$

$$\mathsf{WS}^n_{01}(x_1, \ldots, x_n, c_1) \equiv \mathsf{WS}^{n+1}_{01}(x_1, x_1, x_2, \ldots, x_n, c_1),$$

[1] The preservation condition has been formally verified by a computer program for all entries in the table.

Table 4: Partial operations witnessing non-inclusions in Figure 1.

$\mathsf{C}_2, \mathsf{C}_1, \mathsf{C}_3$	f, f'	$(f_1, \ldots, f_m), (f'_1, \ldots, f'_m)$
$\mathsf{D}_1, \mathsf{I}_2, \mathsf{D}_2$	$xy \vee x\bar{z} \vee y\bar{z}$	$(h_2(x, y, z))$
$\mathsf{D}_1, \mathsf{I}_2, \mathsf{L}_2$	$xy \vee x\bar{z} \vee y\bar{z}$	$(x \oplus y \oplus z)$
$\mathsf{R}_2, \mathsf{I}_2, \mathsf{E}_2$	$x \vee y, x \wedge (y \leftrightarrow z)$	$(x \wedge y), (x \wedge y, x \wedge z)$
$\mathsf{R}_2, \mathsf{I}_2, \mathsf{V}_2$	$x \wedge y, x \wedge (y \leftrightarrow z)$	$(x \vee y), (x \vee y, x \vee z)$
$\mathsf{BF}, \mathsf{I}_2, \mathsf{I}_0$	$x \bar{\wedge} y$	(0)
$\mathsf{BF}, \mathsf{I}_2, \mathsf{I}_1$	$x \bar{\wedge} y$	(1)
$\mathsf{BF}, \mathsf{I}_2, \mathsf{N}_2$	$x \wedge y$	$(\overline{x})$
$\mathsf{R}_1, \mathsf{I}_1, \mathsf{V}_1$	$x \leftrightarrow y$	$(x \vee y)$
$\mathsf{R}_1, \mathsf{I}_1, \mathsf{L}_1$	$x \vee y$	$(x \leftrightarrow y)$
$\mathsf{R}_1, \mathsf{I}_1, \mathsf{E}_1$	$x \vee y, x \leftrightarrow y$	$(x \wedge y), (x \wedge y)$
$\mathsf{R}_1, \mathsf{I}_1, \mathsf{I}$	$x \vee y, x \leftrightarrow y$	$(0), (0)$
$\mathsf{R}_0, \mathsf{I}_0, \mathsf{E}_0$	$x \oplus y$	$(x \wedge y)$
$\mathsf{R}_0, \mathsf{I}_0, \mathsf{L}_0$	$x \wedge y$	$(x \oplus y)$
$\mathsf{R}_0, \mathsf{I}_0, \mathsf{V}_0$	$x \wedge y, x \oplus y$	$(x \vee y), (x \vee y)$
$\mathsf{R}_0, \mathsf{I}_0, \mathsf{I}$	$x \wedge y, x \oplus y$	$(1), (1)$
$\mathsf{M}, \mathsf{I}, \mathsf{V}$	$x \wedge y$	$(x \vee y)$
$\mathsf{M}, \mathsf{I}, \mathsf{E}$	$x \vee y$	$(x \wedge y)$
$\mathsf{M}, \mathsf{I}, \mathsf{N}$	$x \vee y, x \wedge y$	$(1), (1)$
$\mathsf{D}, \mathsf{N}_2, \mathsf{N}$	$xy \vee x\bar{z} \vee \bar{y}\bar{z}$	(1)
$\mathsf{D}, \mathsf{N}_2, \mathsf{L}_3$	$xy \vee x\bar{z} \vee \bar{y}\bar{z}$	$(x \oplus y \oplus z \oplus 1)$
$\mathsf{BF}, \mathsf{N}, \mathsf{L}$	$x \bar{\wedge} y$	$(x \oplus y)$
$\mathsf{BF}, \mathsf{V}_2, \mathsf{V}_1$	$x \bar{\wedge} y$	(1)
$\mathsf{BF}, \mathsf{V}_2, \mathsf{V}_0$	$x \bar{\wedge} y$	(0)
$\mathsf{S}^2_{00}, \mathsf{V}_2, \mathsf{S}_{00}$	$h_2(x, y, z)$	$(x \vee yz, y \vee xz, z \vee xy)$
$\mathsf{S}^2_{01}, \mathsf{V}_1, \mathsf{S}_{01}$	$h_2(x, y, z)$	$(x \vee yz, y \vee xz, z \vee xy)$
$\mathsf{S}^2_{01}, \mathsf{V}_1, \mathsf{V}$	$h_2(x, y, z)$	(0)
$\mathsf{M}_0, \mathsf{V}_0, \mathsf{V}$	$x \wedge y$	(1)
$\mathsf{S}^2_{10}, \mathsf{E}_2, \mathsf{E}_1$	$x \wedge (y \vee z), h_2(x, y, z)$	$(1), (1)$
$\mathsf{S}^2_{10}, \mathsf{E}_2, \mathsf{S}_{10}$	$h_2(x, y, z)$	$(xy \vee xz, yx \vee yz, zx \vee zy)$
$\mathsf{S}^2_{10}, \mathsf{E}_2, \mathsf{E}_0$	$x \wedge (y \vee z), h_2(x, y, z)$	$(0), (0)$
$\mathsf{M}_1, \mathsf{E}_1, \mathsf{E}$	$(x \wedge y)$	(0)
$\mathsf{S}^2_{11}, \mathsf{E}_0, \mathsf{S}_{11}$	$h_2(x, y, z)$	$(xy \vee xz, yx \vee yz, zx \vee zy)$
$\mathsf{S}^2_{11}, \mathsf{E}_0, \mathsf{E}$	$h_2(x, y, z)$	(1)
$\mathsf{BF}, \mathsf{L}_2, \mathsf{L}_0$	$x \bar{\wedge} y$	$(x \oplus y)$
$\mathsf{BF}, \mathsf{L}_2, \mathsf{L}_3$	$x \bar{\wedge} y$	$(y \oplus y \oplus x \oplus 1)$
$\mathsf{BF}, \mathsf{L}_2, \mathsf{L}_1$	$x \bar{\wedge} y$	$(x \leftrightarrow y)$
$\mathsf{D}, \mathsf{L}_3, \mathsf{L}$	$xy \vee x\bar{z} \vee \bar{y}\bar{z}$	(1)
$\mathsf{R}_1, \mathsf{L}_1, \mathsf{L}$	$x \vee y$	(1)
$\mathsf{R}_0, \mathsf{L}_0, \mathsf{L}$	$x \wedge y$	(1)
$\mathsf{D}_1, \mathsf{D}_2, \mathsf{S}^2_{00}$	$xy \vee x\bar{z} \vee y\bar{z}$	$(x \vee yz)$
$\mathsf{BF}, \mathsf{D}_1, \mathsf{D}$	$x \bar{\wedge} y$	$(xy \vee \bar{x}\bar{y})$
$\mathsf{BF}, \mathsf{R}_2, \mathsf{R}_1$	$x \bar{\wedge} y$	$(x \leftrightarrow y)$
$\mathsf{BF}, \mathsf{R}_2, \mathsf{R}_0$	$x \bar{\wedge} y$	$(x \oplus y)$

$$\mathsf{WS}^n_{00}(c_0, x_1, \ldots, x_n, c_1) \equiv \mathsf{WS}^{n+1}_{00}(c_0, x_1, x_1, x_2, \ldots, x_n, c_1),$$

and

$$\mathsf{WS}^n_{02}(c_0, x_1, \ldots, x_n, c_1) \equiv \mathsf{WS}^n_{00}(c_0, c_0, x_1, \ldots, x_n, c_1).$$

For a case $\mathrm{PC} \not\subseteq \mathrm{PC}'$ where inclusion does not hold we provide a partial operation f preserving WC' but not WC. Let f be the unary partial operation $f(1) = 0$. We claim that $f \in \mathrm{PS}^k_{02} \setminus \mathrm{PS}^n_0$, where $n \geq 2$. From Table 2.1 we see that $t[1] = 0$ for every $t \in \mathsf{WS}^k_{02}$, implying that $f(t)$ is always undefined and that f preserves WS^k_{02}. On the other hand, $1^n \in \mathsf{WS}^n_0$ but $0^n \notin \mathsf{WS}^n_0$, where $1^n = (1, \ldots, 1)$ and $0^n = (0, \ldots, 0)$ (both n-ary tuples) implying that $f(1^n) \notin \mathsf{WS}^n_0$. Using similar arguments it can be seen that $f \in \mathrm{Pol}\ S^k_{00} \setminus \mathrm{Pol}\ S^n_0$ and $f \in PS^k_{00} \setminus \mathrm{Pol}\ S^n_{01}$.

For the remaining case we define a binary partial operation f' such that $\mathrm{dom}(f') = \{(0,1), (1,0), (1,1)\}$ and $f'(0,1) = f'(1,0) = 0$, $f'(1,1) = 1$. From Table 2.1 we see that $\mathsf{WS}^k_{01} = \{(0, x_1, \ldots, x_n) | (x_1, \ldots, x_n) \in \{0,1\}^{k-1} \setminus 0^{k-1}\} \cup \{1^{k+1}\}$. This means that $f'(s,t)$ is defined for $s, t \in \mathsf{WS}^k_{01}$ only if there does not exist $i \in \{1, \ldots, k\}$ such that $s[i] = t[i] = 0$. Hence, if $f(s,t)$ is defined, then at least one of s and t is equal to 1^n. If $s = t = 1^k$, then $f'(s,t) = 1^k$, and if $s \neq t$, then from the definition of f' it must be the case that $f'(s,t) = s$ assuming $t = 1^k$ (the case when $s = 1^k$ is symmetric). This proves that $f' \in \mathrm{PS}^k_{01}$. On the other hand, there exists $u, v \in \mathsf{WS}^n_0$ such that $u[i] \oplus v[i] = 1$ for $i \in \{1, \ldots, n\}$, and such that $u[n+1] = v[n+1] = 1$. This implies that $f'(u,v)$ is defined and returns a tuple w where $w[i] = 0$ for $i \in \{1, \ldots, n\}$, and where $w[n+1] = 1$. But then $w \notin \mathsf{WS}^n_0$. Hence, we conclude that f' preserves WS^k_{01} but not WS^n_0. $\square$

Lemma 8 is also valid for $\mathrm{PS}_0, \mathrm{PS}_{02}, \mathrm{PS}_{01}, \mathrm{PS}_{00}$, and can be proved for the dual clones in Figure 1 using entirely analogous arguments. Finally, by combining the results in this section we may now prove the main result of the paper.

Theorem 9. *Let C, C' be two Boolean clones. Then $\mathrm{PC} \subset \mathrm{PC}'$ if and only if there exists a path consisting of upward edges connecting PC to PC' in Figure 1.*

Proof. All positive inclusions in Figure 1 follow from Table 3 and Lemma 8. Assume that $\mathrm{PC} \not\subseteq \mathrm{PC}'$ according to Figure 1 but that $\mathsf{C} \subset \mathsf{C}'$. If $\mathsf{S}_{00} \subseteq \mathsf{C} \subseteq \mathsf{S}^2_0$, or $\mathsf{S}_{10} \subseteq \mathsf{C} \subseteq \mathsf{S}^2_1$, then the non-inclusion follows from Lemma 8. Otherwise there exists an entry $\mathsf{C}_2, \mathsf{C}, \mathsf{C}_3$ in Table 3 such that $\mathsf{C}_3 \in Cov(\mathsf{C})$, $\mathsf{C}_3 \subseteq C'$, $f, f' \in \mathsf{C}_2$, $g^f_{f_1,\ldots,f_m}, g^{f'}_{f'_1,\ldots,f'_m} \in \mathrm{PC}$ and $f_1, f'_1, \ldots, f_m, f'_m \in \mathsf{C}_3$ such that $[\{g^f_{f_1,\ldots,f_m}, g^{f'}_{f'_1,\ldots,f'_m}, f_1, \ldots, f_m, f'_1, \ldots, f'_m\}]_s \subseteq [\mathsf{PC} \cup \mathsf{C}_3]_s \cap \mathsf{BF} \not\subseteq \mathsf{C}'$. Theorem 6 then gives the desired result that $\mathrm{PC} \not\subseteq \mathrm{PC}'$. $\square$

Theorem 9 can also be used to prove a stronger result for the two weak bases $\mathrm{P}\mathsf{I}_2$ and PD_1. These two weak bases have appeared in the literature before: $\mathrm{P}\mathsf{I}_2$ has been proved to result in the "easiest NP-complete satisfiability problem" [10], and PD_1 is known to be a submaximal strong partial clone, i.e., is covered by a maximal strong partial clone. We can then prove that $\mathrm{P}\mathsf{I}_2$ is uniquely covered by PD_1.

Theorem 10. *$\mathrm{P}\mathsf{I}_2$ is covered by PD_1 and by no other Boolean strong partial clone.*

Proof. According to Theorem 9 and Figure 1, $\mathrm{P}\mathsf{I}_2 \subset \mathrm{PD}_1 \subset \mathrm{P}\mathsf{R}_2 \subset \mathrm{P}\mathsf{BF}$, and there does not exist any other clone C such that $\mathrm{P}\mathsf{I}_2 \subset \mathrm{PC}$. Assume, with the aim of reaching a contradiction, that there exists $\mathrm{pPol}(\Gamma)$ covering $\mathrm{P}\mathsf{I}_2$ but which is distinct from PD_1, and let $\mathrm{Pol}(\Gamma) = \mathsf{C}$. First, we observe that it must then be the case that $\mathsf{I}_2 \subseteq \mathsf{C} \subseteq \mathsf{D}_1$, since $\mathrm{P}\mathsf{I}_2$ is incomparable to $\mathrm{PC} \supseteq \mathrm{pPol}(\Gamma)$

otherwise. Second, $\mathsf{I}_2 \subset \mathsf{C} \subset \mathsf{D}_1$ cannot happen either since $\mathrm{pPol}(\Gamma) \subseteq \mathsf{PC}$, and we have already established that $\mathsf{PI}_2 \not\subseteq \mathsf{PC}$. Hence, either $\mathsf{C} = \mathsf{I}_2$ or $\mathsf{C} = \mathsf{D}_1$. The first case cannot occur since it would imply that WI_2 is not a weak base of II_2, which leaves only the case when $\mathsf{C} = \mathsf{D}_1$. According to Haddad & Simons [9] this can only occur if $\mathrm{pPol}(\Gamma)$ is equal to $\mathrm{pPol}(\{\mathsf{WD}_1, \mathrm{F}\})$, $\mathrm{pPol}(\{\mathsf{WD}_1, \mathrm{T}\})$, $\mathrm{pPol}(\{\mathrm{Neq} \times \mathrm{T}\})$, $\mathrm{pPol}(\{\mathrm{Neq} \times \mathrm{F}\})$, or $\mathrm{pPol}(\{\mathrm{Neq}, \mathrm{F} \times \mathrm{T}\})$. For each of these cases it is then readily verified that $\mathsf{PI}_2 \not\subseteq \mathrm{pPol}(\Gamma)$. For example, if $\Gamma = \{\mathsf{WD}_1, \mathrm{F}\}$ then the partial operation $f(0) = 1$ preserves WI_2 but does not preserve $\mathrm{F} \in \Gamma$. $\square$

4 Concluding Remarks

In this paper we have fully described the inclusion structure of Boolean weak bases. An interesting continuation, especially in light of Theorem 10, is to verify, or disprove, that an inclusion between PC and PC' in Figure 1 also implies that PC is covered by PC'. Another suitable topic is to study weak bases over arbitrary finite domains. In this setting we cannot hope for a complete classification akin to Figure 1, but even partial results could be of interest. For example, given a minimal clone C over a finite domain D, is it possible to describe a weak base of $\mathrm{Inv}(\mathsf{C})$?

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