A New Characterization of Restriction-Closed Hyperclones

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Abstract

A *hyperoperation* is a mapping from a domain to the powerset of the domain. Hyperoperations can be composed together to form new hyperoperations, and the resulting sets are called *hyperclones*. In this paper we study the lattice of *restriction-closed* hyperclones over finite domains. Such hyperclones form a natural subclass of hyperclones but have received comparably little attention. We give a complete description of restriction-closed hyperclones, relative to the clone lattice, and also outline some important open questions to resolve when studying hyperclones over partially defined operations.

1 Introduction

A *clone* is a set of operations over a given domain which (1) contains all operations returning the value of a fixed argument, and (2) is closed under composition. Clones occur naturally in universal algebra since a clone generated by a set of operations F may be viewed as the *term algebra* generated by F. From an applied point of view clones are important in computational complexity due to their connection to e.g. *constraint satisfaction problems* (CSPs) [2, 14]. For example, computational properties of CSPs can be rephrased as properties of clones, which in turn has a strong connection to sets of relations closed under existentially quantified conjunctive formulas with equality, *primitive positive definitions* (pp-definitions).

A *hyperclone*, or a *clone of multifunctions*, is a generalisation of a clone where the operations may return sets instead of single elements

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from the domain [13, 11]. Despite being a natural generalisation of clones, comparably little is known about hyperclones, except for a handful of completeness results [8, 3, 9]. One reason for this discrepancy, or at least a complicating factor, is that the relational counterparts to hyperclones are more complicated and less natural than closure under pp-definitions (cf., Colic et al [4]). However, there exists a restriction of hyperclones admitting a natural relational description in terms of existentially quantified conjunctive formulas (without equality), *equality-free primitive positive definition* (efpp-definitions). The hyperclones resulting from this correspondence are then *restriction-closed*, meaning that any restriction of a hyperoperation in the hyperclone also belongs to it (see Section 2 for the definition of a restriction). Efpp-definitions can e.g. be useful when the presence of equality constraints affect the structure of a CSP instance, which has been used to relate the complexity of *degree-bounded* CSPs to a complexity theoretical conjecture known as the *exponential-time hypothesis* [6]. However, despite the potential of applications in computational complexity, and a natural connection to efpp-definitions, very little is known about the restriction-closed case. One exception is Romov [11] who proved that each Boolean clone splits up into at most two restriction-closed hyperclones, and classified the Boolean clones according to this property. In contrast, the lattice of hyperclones is uncountably infinite even in the Boolean domain [7], suggesting a large difference to restriction-closed hyperclones.

In this paper we continue the study of restriction-closed hyperclones, with a particular focus of describing the set, sometimes called an *interval*, of restriction-closed hyperclones corresponding to a given clone. After having defined the most important preliminaries (in Section 2) we in Section 3 tackle this problem for arbitrary finite domains. We begin by outlining various relational simplifications based on efpp-definability in Section 3.1, and then apply this in Section 3.2 to give a precise description of intervals of restriction-closed hyperclones. In particular, our classification implies that each clone splits into a finite number of restriction-closed hyperclones, meaning that the lattice of restriction-closed hyperclones is not significantly more complicated than the clone lattice. In Section 4 we also consider the corresponding classification question for restriction-closed hyperclones over *partial hyperoperations*. Although partial hyperoperations have been studied before, little is known for the restricted-closed case. Here, our results are more example driven, and we give examples where the corresponding intervals differ from the total case. Last, we wrap up the paper by highlighting future research directions in Section 5.

2 Preliminaries

Throughout, we let $D \subseteq \mathbb{N}$ denote a fixed set of elements, usually called a *domain* or a *universe*.

2.1 Relations

If $R \subseteq D^n$ is an n-ary relation then we let $\mathrm{ar}(R) = n$ denote its arity. For an n-tuple $t = (d_1, \ldots, d_n) \in D^n$ and $1 \leq i \leq n$ we let $t[i]$ be the ith element of t, i.e., $t[i] = d_i$. More generally, if $i_1, \ldots, i_k \in \{1, \ldots, n\}$, $1 \leq k \leq n$, are mutually distinct indices of an n-ary relation R we write $\mathrm{Proj}_{i_1,\ldots,i_k}(R)$ for the *projection* of R onto $i_1, \ldots, i_k$, i.e.,

$$\mathrm{Proj}_{i_1,\ldots,i_k}(R) = \{(t[i_1], \ldots, t[i_k]) \mid t \in R\}.$$

We write $\mathcal{R}_D$ for the set of all relations over a given domain D. A relation $R \in \mathcal{R}_D$ is said to be *irredundant* if there does not exist distinct indices $i, j \in \{1, \ldots, ar(R)\}$ such that $t[i] = t[j]$ for each $t \in R$, and we write $\mathcal{IR}_D$ for the set of all irredundant relations over D. If $\varphi(x_1, \ldots, x_n)$ is a first-order formula with free variables $x_1, \ldots, x_n$ then we write $R(x_1, \ldots, x_n) \equiv \varphi(x_1, \ldots, x_n)$ to define the relation

$$R = \{(f(x_1), \ldots, f(x_n)) \mid f \text{ is a model of } \varphi(x_1, \ldots, x_n)\}.$$

Write $\mathrm{Eq}_D = \{(x, x) \mid x \in D\}$ for the equality relation over D. Let $\mathsf{f} = \emptyset$ (viewed as a unary relation) and $\mathsf{t} = D$, and observe that $\mathsf{f}(x)$ is always false and that $\mathsf{t}(x)$ is always true (and equivalent to $\mathrm{Eq}_D(x, x)$).

Definition 1. *An n-ary relation R over D is said to have a* primitive positive definition *(pp-definition) over $\Gamma \subseteq \mathcal{R}_D$ if*

$$R(x_1, \ldots, x_n) \equiv \exists x_{n+1}, \ldots, x_{n+n'} : \quad R_1(\mathbf{x}_1) \wedge \ldots \wedge R_m(\mathbf{x}_m),$$

where each $R_i \in \Gamma \cup \{\mathrm{Eq}_D, \mathsf{f}, \mathsf{t}\}$ and each $\mathbf{x}_i$ is an $\mathrm{ar}(R_i)$-ary tuple of variables over $x_1, \ldots, x_n$, $x_{n+1}, \ldots, x_{n+n'}$.

If, in addition, each $R_i \in \Gamma \cup \{\mathsf{f}, \mathsf{t}\}$ then we say that R has an *equality-free primitive positive definition* (efpp-definition) over Γ. Let $\langle \Gamma \rangle_{\neq}$ be the set of efpp-definable relations over Γ and $\langle \Gamma \rangle$ be the set of pp-definable relations over Γ. Note that $\langle \Gamma \rangle_{\neq} \subseteq \langle \Gamma \rangle$ for each set $\Gamma \subseteq \mathcal{R}_D$ but that equality does not necessarily hold.

2.2 Operations

We now consider the functional counterparts to pp- and efpp-definitions from the preceding section. Let $\mathcal{P}(D)$ be the powerset of a set D.

A k-ary function $f\colon D^k \to \mathcal{P}(D) \setminus \{\emptyset\}$ is said to be a *hyperoperation*, or *multifunction*, over D. If R is an n-ary relation over D then a k-ary hyperoperation f is said to *preserve* R if $f(t_1, \ldots, t_k) \subseteq R$ for a sequence of tuples $t_1, \ldots, t_k \in R$, where

$$f(t_1, \ldots, t_k) = f(t_1[1], \ldots, t_k[1]) \times \ldots \times f(t_1[n], \ldots, t_k[n]).$$

If this holds we also say that f is a *hyperpolymorphism* of R, or that R is *invariant* under f. This notion easily generalises to sets of relations and we say that f preserves $\Gamma \subseteq \mathcal{R}_D$ if f preserves each relation in Γ. If f and g are two k-ary hyperoperations we say that g is a *restriction* of f if $g(x_1, \ldots, x_k) \subseteq f(x_1, \ldots, x_k)$ for all $x_1, \ldots, x_k \in D$. It is well-known, and easy to verify, that if f preserves a set of relations Γ then each restriction g of f also preserves Γ. If F is a set of hyperoperations where $f \in F$ implies that $g \in F$ for each restriction g of f, then F is said to be *restriction-closed*.

A hyperoperation $f\colon D^k \to \mathcal{P}(D) \setminus \{\emptyset\}$ is said to be *elementary* if $|f(x_1, \ldots, x_k)| = 1$ for all sequences of arguments $x_1, \ldots, x_k \in D$, and for simplicity we typically do not make a sharp distinction between elementary hyperoperations and *operations* from D^k to D. The set of all hyperpolymorphisms of a set of relations Γ is written as $\mathrm{hPol}(\Gamma)$, and we similarly let $\mathrm{Pol}(\Gamma)$ be the set of all polymorphisms, i.e., the set of all elementary hyperpolymorphisms of Γ. Sets of the form $\mathrm{Pol}(\Gamma)$ are called *clones*, and sets of the form $\mathrm{hPol}(\Gamma)$ restriction-closed *hyperclones* or *clones of multifunctions*.

Dually, we let $\mathrm{Inv}(F)$ be the set of relations invariant under the set of (hyper)operations F. It is then known that $\langle\Gamma\rangle_{\neq} = \mathrm{Inv}(\mathrm{hPol}(\Gamma))$ and that $\langle\Gamma\rangle = \mathrm{Inv}(\mathrm{Pol}(\Gamma))$. Morevoer, hyperoperations, operations, efpp-definitions, and pp-definitions can be related as follows.

Theorem 2 ([5, 10])**.** *Let Γ and Δ be two sets of relations. Then (1) $\Gamma \subseteq \langle\Delta\rangle_{\neq}$ if and only if $\mathrm{hPol}(\Delta) \subseteq \mathrm{hPol}(\Gamma)$ and (2) $\Gamma \subseteq \langle\Delta\rangle$ if and only if $\mathrm{Pol}(\Delta) \subseteq \mathrm{Pol}(\Gamma)$.*

The definition of a hyperoperation is easy to tweak into the setting of partially defined operations: say that f is a *partial hyperoperation* if $f\colon D^n \to \mathcal{P}(D)$. Here, $f(x_1, \ldots, x_n) = \emptyset$ is treated as an undefined value (recall that the definition of a hyperoperation prohibited the empty set in the image of a hyperoperation). The basic definitions immediately carry over to this setting, and we let $\mathrm{hpPol}(\Gamma)$ be the set of all partial hyperpolymorphisms of a set of relations Γ. Such sets have been referred to as *restriction-closed partial hyperclones* [12]. On the relational side the corresponding closure operator is then closure under efpp-definitions without existential quantification, and we write $\langle\Gamma\rangle_{\neq\nexists}$ for the smallest set of relations containing Γ which is closed under such definitions. Similarly to the total case we say that a partial hyperoperation is elementary if its image consists only of singleton sets and $\emptyset$, in which case it is simply called a *partial operation*, and a partial, elementary hyperpolymorphism of Γ is called a *partial polymorphism* of Γ. We write $\mathrm{pPol}(\Gamma)$ for the set of all partial polymorphisms of Γ, a *strong partial clone*, and let $\langle\Gamma\rangle_{\nexists}$ be the smallest set of relations containing Γ closed under pp-definitions without existential quantification. It is then known that $\langle\Gamma\rangle_{\neq\nexists} = \mathrm{Inv}(\mathrm{hpPol}(\Gamma))$, that $\langle\Gamma\rangle_{\nexists} = \mathrm{Inv}(\mathrm{pPol}(\Gamma))$, and that the operators can be related as follows.

Theorem 3 ([10, 12])**.** *Let Γ and Δ be two sets of relations. Then (1) $\Gamma \subseteq \langle\Delta\rangle_{\neq\nexists}$ if and only if $\mathrm{hpPol}(\Delta) \subseteq \mathrm{hpPol}(\Gamma)$ and (2) $\Gamma \subseteq \langle\Delta\rangle_{\nexists}$ if and only if $\mathrm{pPol}(\Delta) \subseteq \mathrm{pPol}(\Gamma)$.*

Last, let us remark that clones, hyperclones, and their partial variants can all be defined purely from the functional side by defining a suitable notion of functional composition. However, in this paper we settle with relational definitions since most of our results are proven in this setting.

3 The Lattice of Restriction-Closed Hyperclones

In this section we tackle the main problem in the paper, i.e., that of describing the restriction-closed hyperclones over a fixed, finite domain. In Section 3.2 we will see that this question to a great extent can be simplified by first fixing a clone C and then concentrate on describing the restriction-closed hyperclones whose set of elementary operations equals C. However, let us first begin by establishing some fundamental properties of efpp-definitions, which will greatly simplify the forthcoming proofs.

3.1 Properties of Efpp-Definitions

For an n-ary relation R over D we let

$$\mathcal{E}_R = \{(i,j) \in \{1,\ldots,n\}^2 \mid i \neq j, \forall t \in R \colon t[i] = t[j]\}.$$

Thus, $\mathcal{E}_R$ returns a set of indices where each pair witnesses an argument in R which is not irredundant.

Example 1. *For any irredundant relation R we have that $\mathcal{E}_R = \emptyset$. For the equality relation Eq_D we instead get $\mathcal{E}_{\mathrm{Eq}_D} = \{(1,2),(2,1)\}$.*

Lemma 4. *If $R \in \langle\Gamma\rangle$ then there exists a pp-definition of R over Γ*

such that

$$R(x_1, \ldots, x_n) \equiv \exists x_{n+1}, \ldots, x_{n+m}\colon \varphi(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) \wedge \bigwedge_{(i,j) \in \mathcal{E}_R} \mathrm{Eq}_D(x_i, x_j)$$

where $\varphi(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ *is equality-free.*

Proof. Let $\mathcal{E}_R = \{(i_1, j_1), \ldots, (i_k, j_k)\}$, and take a pp-definition

$$R(x_1, \ldots, x_n) \equiv \exists x_{n+1}, \ldots, x_{n+m}\colon \varphi(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) \wedge \bigwedge_{(i,j) \in I} \mathrm{Eq}_D(x_i, x_j)$$

for a set $I \subseteq \{1, \ldots, n, n+1, \ldots, n+m\}^2$, where

$$\varphi(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$$

is equality-free. We first observe that there cannot exist $(i, j) \in I \cap \{1, \ldots, n\}^2$ such that $(i, j) \notin \{(i_1, j_1), \ldots, (i_k, j_k)\}$ since this contradicts the definition of the latter set. Take $(i, j) \in I$ and assume first that $i = j$. In this case we replace the constraint $\mathrm{Eq}_D(x_i, x_i)$ with the constraint $\mathtt{t}(x_i)$. Next, assume without loss of generality that $i < j$, where $j \geq n+1$. Then there exists an equality constraint $\mathrm{Eq}_D(x_i, x_j)$ in the pp-definition of R, and we can then obtain a shorter pp-definition by (1) removing the equality constraint, (2) removing x_j from $\exists x_{n+1}, \ldots, x_{n+m}$, and (3) repeatedly replacing each occurrence of x_j with x_i. If we repeat this for each $(i, j) \in I$ where $(i, j) \notin \{(i_1, j_1), \ldots, (i_k, j_k)\}$ we will obtain a pp-definition of R over Γ satisfying the form stated in the lemma. □

The following useful lemma, roughly stating that efpp-definitions have the same expressive strength as pp-definitions if the relation in question is irredundant, is now a straightforward consequence

Lemma 5. *Let* $R \in \mathcal{IR}_D$ *and let* $\Gamma \subseteq \mathcal{R}_D$*. If* $R \in \langle \Gamma \rangle$ *then* $R \in \langle \Gamma \rangle_{\neq}$*.*

Proof. Assume that $R \in \langle \Gamma \rangle \cap \mathcal{IR}_D$ is n-ary. Lemma 4 then directly implies that

$$R(x_1, \ldots, x_n) \equiv \exists x_{n+1}, \ldots, x_{n+m}\colon \varphi(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}),$$

since the set of indices $\mathcal{E}_R$ is empty whenever the relation R is irredundant. □

3.2 Intervals of Restriction-Closed Hyperclones

Our main technical simplification is to study sets of restriction-closed hyperclones corresponding to a given clone C. Since many of our technical lemmas are stated via relational tools we therefore begin by defining the corresponding sets from the relational side.

Definition 6. *Let C be a clone over D. We let $\mathcal{L}_{|\mathrm{Inv}(C)} = \{\langle\Gamma\rangle_{\neq} \mid \Gamma \subseteq \mathcal{R}_D, \langle\Gamma\rangle = \mathrm{Inv}(C)\}$.*

We can then readily prove that $\mathcal{L}_{|\mathrm{Inv}(C)}$ has a greatest and least element.

Theorem 7. *Let C be a clone over D. Then the greatest element in $\mathcal{L}_{|\mathrm{Inv}(C)}$ is $\mathrm{Inv}(C)$, and the least element is $\langle \mathrm{Inv}(C) \cap \mathcal{IR}_D\rangle_{\neq}$.*

Proof. $\mathrm{Inv}(C)$ is trivially the greatest element in $\mathcal{L}_{|\mathrm{Inv}(C)}$. To see that $\langle \mathrm{Inv}(C) \cap \mathcal{IR}_D\rangle_{\neq}$ is the least element, take an arbitrary $\langle\Gamma\rangle_{\neq} \in \mathcal{L}_{|\mathrm{Inv}(C)}$. It follows that $R \in \langle\Gamma\rangle_{\neq}$ for each $R \in \mathrm{Inv}(C) \cap \mathcal{IR}_D$ (by Lemma 5) from which we conclude that $\langle \mathrm{Inv}(C) \cap \mathcal{IR}_D\rangle_{\neq} \subseteq \langle\Gamma\rangle_{\neq}$. ☐

Our principal task is now to describe the sets $\mathcal{L}_{|\mathrm{Inv}(C)}$ for each clone C, which we will henceforth refer to as *intervals*.

Theorem 8. *Let C be a clone over a finite D. For each $\langle\Gamma\rangle_{\neq} \in \mathcal{L}_{|\mathrm{Inv}(C)}$ there exists $D_1, \ldots, D_d \subseteq D$ such that $\langle\Gamma\rangle_{\neq} = \langle(\mathrm{Inv}(C) \cap \mathcal{IR}_D) \cup \{\mathrm{Eq}_{D_1}, \ldots, \mathrm{Eq}_{D_d}\}\rangle_{\neq}$.*

Proof. Let $\langle\Gamma\rangle_{\neq} \in \mathcal{L}_{|\mathrm{Inv}(C)}$, and for simplicity of notation let $\Delta = \mathrm{Inv}(C) \cap \mathcal{IR}_D$. We claim that $\langle\Gamma\rangle_{\neq} = \langle\Delta \cup \{\mathrm{Eq}_{D_1}, \ldots, \mathrm{Eq}_{D_d}\}\rangle_{\neq}$ for some $D_1, \ldots, D_d \subseteq D$. First, partition Γ into two disjoint sets Γ_1 and Γ_2 such that $\Gamma_1 \cup \Gamma_2 = \Gamma$, $\Gamma_1 \subseteq \langle\Delta\rangle_{\neq}$, and $R \notin \langle\Delta\rangle_{\neq}$ for each $R \in \Gamma_2$. In other words each relation in Γ_1 is efpp-definable by Δ, but no relation in Γ_2 is efpp-definable by Δ. For each relation $R \in \Gamma_2$ then take the pp-definition

$$R(x_1, \ldots, x_n) \equiv \exists x_{n+1}, \ldots, x_{n+m}\colon \varphi(x_1, \ldots, x_{n+m}) \wedge \mathrm{Eq}_D(x_{i_1}, x_{j_1}) \wedge \ldots \wedge \mathrm{Eq}_D(x_{i_k}, x_{j_k})$$

of R over Δ from Lemma 4, where $i_1, j_1, \ldots, i_k, j_k \in \{1, \ldots, n\}$ and $\varphi(x_1, \ldots, x_{n+m})$ is equality-free. Let $D_{i_1}, \ldots, D_{i_k}$ be defined so that $D_{i_1} = \mathrm{Proj}_{i_1}(R), \ldots, D_{i_k} = \mathrm{Proj}_{i_k}(R)$, and consider the relations $\mathrm{Eq}_{D_{i_1}}, \ldots, \mathrm{Eq}_{D_{i_k}}$. Then we first observe that

$$R(x_1, \ldots, x_n) \equiv \exists x_{n+1}, \ldots, x_{n+m}\colon \varphi(x_1, \ldots, x_{n+m}) \wedge \mathrm{Eq}_{D_{i_1}}(x_{i_1}, x_{j_1}) \wedge \ldots \wedge \mathrm{Eq}_{D_{i_k}}(x_{i_k}, x_{i_k}),$$

implying that $R \in \langle \Delta \cup \{\mathrm{Eq}_{D_{i_1}}, \ldots, \mathrm{Eq}_{D_{i_k}}\}\rangle_{\neq}$. Second, also note that each $\mathrm{Eq}_{D_{i_l}} = \mathrm{Proj}_{i_l, j_l}(R)$, $1 \leq l \leq k$, implying that each $\mathrm{Eq}_{D_{i_l, j_l}} \in \langle\{R\}\rangle_{\neq} \subseteq \langle\Gamma\rangle_{\neq}$. Let $E \subseteq \{\mathrm{Eq}_{D_i} \mid D_i \subseteq D\}$ be the set of equality relations resulting from repeating this for each $R \in \Gamma_2$. From the construction of E it follows that $\langle \Delta \cup E\rangle_{\neq} = \langle\Gamma\rangle_{\neq}$ and that E is a finite set since D is finite. □

In particular, Theorem 8 implies that $\mathcal{L}_{|\mathrm{Inv}(C)}$ is finite for *every* clone C over a finite domain D, since there only exists a finite number of subsets of D. We can make the description of $\mathcal{L}_{|\mathrm{Inv}(C)}$ even more precise by also describing the inclusion structure of its elements, but before we turn to this problem we first need to describe the expressive strength of $\langle(\mathrm{Inv}(C) \cap \mathcal{IR}_D) \cup E\rangle_{\neq} \in \mathcal{L}_{|\mathrm{Inv}(C)}$.

Lemma 9. *Let* $\mathrm{Inv}(C)$ *be a co-clone over* D. *Let* $D_1, \ldots, D_k \subseteq D$ *be the unary relations such that*

1) $\mathrm{Eq}_{D_i} \notin \langle \mathrm{Inv}(C) \cap \mathcal{IR}_D\rangle_{\neq}$, *and*

2) $D_i \in \mathrm{Inv}(C)$ *for each* $1 \leq i \leq k$.

Let $E \subseteq \{\mathrm{Eq}_{D_1}, \ldots, \mathrm{Eq}_{D_k}\}$. *Then* $\mathrm{Eq}_{D_i} \in \langle(\mathrm{Inv}(C) \cap \mathcal{IR}_D) \cup E\rangle_{\neq}$, $1 \leq i \leq k$, *if and only if there exists* $\mathrm{Eq}_{D_j} \in E$ *such that* $D_i \subseteq D_j$.

Proof. Let $\Delta = \mathrm{Inv}(C) \cap \mathcal{IR}_D$. First, let $\mathrm{Eq}_{D_j} \in E$, pick a subset $D_i \subseteq D_j$ where $D_i \in \mathrm{Inv}(C)$. Since D_i by definition is irredundant it is efpp-definable by Δ (from Lemma 5), and it follows that $\mathrm{Eq}_{D_i}(x_1, x_2) \equiv \mathrm{Eq}_{D_j}(x_1, x_2) \wedge D_i(x_1)$.

Second, assume that $\mathrm{Eq}_{D_i} \in \langle \Delta \cup E\rangle_{\neq}$, but that there does not exist any $\mathrm{Eq}_{D_j} \in E$ such that $\mathrm{Eq}_{D_i} \subseteq \mathrm{Eq}_{D_j}$ (again, $D_i \in \langle\Delta\rangle_{\neq}$ since D_i is irredundant). Let $\mathrm{Eq}_{D_i}(x_1, x_2) \equiv \exists y_1, \ldots, y_n \colon \varphi(x_1, x_2, y_1, \ldots, y_n)$ be an efpp-definition of Eq_{D_i} over $\Delta \cup E$. Pick a constraint $R(z_1, \ldots, z_m)$ in $\varphi(x_1, x_2, y_1, \ldots, y_n)$, where each $z_i \in \{x_1, x_2, y_1, \ldots y_n\}$. Assume that $R = \mathrm{Eq}_{D_j}$ for some $\mathrm{Eq}_{D_j} \in E$ (in which case $m = 2$). First, assume that $z_1 \in \{x_1, x_2\}$ or that $z_2 \in \{x_1, x_2\}$. Since we assumed that Eq_{D_i} is not included in Eq_{D_j} for any $\mathrm{Eq}_{D_j} \in E$, there exists $d \in D_i$ where $d \notin D_j$. But then the constraint $\mathrm{Eq}_{D_j}(z_1, z_2)$ violates the assumption that $\exists y_1, \ldots, y_n \colon \varphi(x_1, x_2, y_1, \ldots, y_n)$ defines Eq_{D_i}. Hence, $z_1, z_2 \in \{y_1, \ldots, y_m\}$. But then we may simply remove z_2, replace each occurrence by z_1, and replace the constraint $\mathrm{Eq}_{D_j}(z_1, z_2)$ by $D_j(z_1)$, which is efpp-definable over Δ via Lemma 5. Note also that the case when $z_1 = z_2$ may be handled simply by replacing the constraint by $D_i(z_1)$. If we repeat this for each constraint $R(z_1, \ldots, z_m)$ in $\varphi(x_1, x_2, y_1, \ldots, y_n)$ where $R \in E$ then we will obtain an efpp-definition of Eq_{D_i} over Δ, which contradicts the original assumption $\mathrm{Eq}_{D_i} \notin \langle\Delta\rangle_{\neq}$. □

Equipped with Lemma 9 the description of $\mathcal{L}_{|\mathrm{Inv}(C)}$ is now straightforward.

Theorem 10. *Let* $\mathrm{Inv}(C)$ *be a co-clone over* D *and let* $\Delta = \mathrm{Inv}(C) \cap \mathcal{IR}_D$. *Let* $D_1, \ldots, D_k \subseteq D$ *be the unary relations such that (1)* $\mathrm{Eq}_{D_i} \notin \langle\Delta\rangle_{\neq}$ *and (2)* $D_i \in \mathrm{Inv}(C)$ *for each* $1 \leq i \leq k$. *Then, for* $E_1, E_2 \subseteq \{\mathrm{Eq}_{D_1}, \ldots, \mathrm{Eq}_{D_k}\}$,

1) $\langle\Delta \cup E_1\rangle_{\neq} \subseteq \langle\Delta \cup E_2\rangle_{\neq}$ *if for each* $\mathrm{Eq}_{D_i} \in E_1$ *there exists* $\mathrm{Eq}_{D_j} \in E_2$ *such that* $D_i \subseteq D_j$, *and*
2) $\langle\Delta \cup E_1\rangle_{\neq} \not\subseteq \langle\Delta \cup E_2\rangle_{\neq}$ *otherwise.*

Proof. First, assume that there exists $\mathrm{Eq}_{D_i} \in E_1$ but no $\mathrm{Eq}_{D_j} \in E_2$ such that $D_i \subseteq D_j$. It follows that $\mathrm{Eq}_{D_i} \notin \langle\Delta \cup E_2\rangle_{\neq}$ via Lemma 9, and $\langle\Delta \cup E_1\rangle_{\neq}$ cannot be included in $\langle\Delta \cup E_2\rangle_{\neq}$. Similarly, Lemma 9 also implies that if for each $\mathrm{Eq}_{D_i} \in E_1$ there exists $\mathrm{Eq}_{D_j} \in E_2$ where $D_i \subseteq D_j$, then $\mathrm{Eq}_{D_i} \in \langle\Delta \cup E_2\rangle_{\neq}$. Hence, $\langle\Delta \cup E_1\rangle_{\neq} \subseteq \langle\Delta \cup E_2\rangle_{\neq}$. □

The combination of Theorem 8 and Theorem 10 gives a complete description of $\mathcal{L}_{|\mathrm{Inv}(C)}$, with the caveat that the relations $D_1, \ldots, D_k \subseteq D$ where $\mathrm{Eq}_{D_1}, \ldots, \mathrm{Eq}_{D_k} \notin \langle\mathrm{Inv}(C) \cap \mathcal{IR}_D\rangle_{\neq}$ need to be provided. While we have not been able to isolate a general criterion for this, we may at least reformulate this property using hyperpolymorphisms.

Definition 11. *A hyperoperation* $h\colon D^k \to \mathcal{P}(D) \setminus \{\emptyset\}$ *is said to be* D'-elementary *for a set* $D' \subseteq D$ *if* $|h(x_1, \ldots, x_k)| = 1$ *for all* $x_1, \ldots, x_k \in D'$.

The existence of $\mathrm{Eq}_{D_i} \in \langle\mathrm{Inv}(C) \cap \mathcal{IR}_D\rangle_{\neq}$ may then be related to the existence of D'-elementary hyperoperations, as follows.

Theorem 12. *Let* Γ *be a set of relations over* D *and let* $D' \subseteq D$ *be a unary relation such that* $D' \in \langle\Gamma\rangle$. *Then* $\mathrm{Eq}_{D'} \in \langle\Gamma\rangle_{\neq}$ *if and only if each* $h \in \mathrm{hPol}(\Gamma)$ *is* D'*-elementary.*

Proof. First, assume that $\mathrm{Eq}_{D'} \notin \langle\Gamma\rangle_{\neq}$. Hence, there exists at least one hyperoperation $h \in \mathrm{hPol}(\Gamma)$ which does not preserve $\mathrm{Eq}_{D'}$. Thus, if we let n be the arity of h, then there exists $t_1, \ldots, t_n \in \mathrm{Eq}_{D'}$ such that $h(t_1, \ldots, t_n) \notin \mathrm{Eq}_{D'}$. First, assume that

$$|h(t_1[1], \ldots, t_n[1])| = |h(t_1[2], \ldots, t_n[2])| = 1.$$

It follows that

$$h(t_1[1], \ldots, t_n[1]) = h(t_1[2], \ldots, t_n[2]) \notin D',$$

and that h does not preserve D'. This contradicts the assumption that $D' \in \langle\Gamma\rangle$ since the fact that D' is irredundant also implies that $D' \in \langle\Gamma\rangle_{\neq}$, and that D' must be preserved by every $h \in \mathrm{hPol}(\Gamma)$. Hence, if

$h(t_1, \ldots, t_n) \notin \mathrm{Eq}_{D'}$ it must be the case that either $|h(t_1[1], \ldots, t_n[1])| > 1$ or that $|h(t_1[2], \ldots, t_n[2])| > 1$.

Second, assume that $\mathrm{Eq}_{D'} \in \langle \Gamma \rangle_{\neq}$, and consider an application $h(t_1, \ldots, t_n)$ for $t_1, \ldots, t_n \in \mathrm{Eq}_{D'}$ for an n-ary $h \in \mathrm{hPol}(\Gamma)$. If $h(t_1[1], \ldots, t_n[1]) = X$ for $|X| > 1$, then X contains at least two distinct elements d_1, d_2. Pick one additional element d_3 from $h(t_1[2], \ldots, t_n[2])$. Then $h(t_1, \ldots, t_n) = h(t_1[1], \ldots, t_n[1]) \times h(t_1[2], \ldots, t_n[2]) = X \times h(t_1[2], \ldots, t_n[2]) \not\subseteq \mathrm{Eq}_{D'}$ since (d_1, d_3) and (d_2, d_3) are both included in $h(t_1[1], \ldots, t_n[1]) \times h(t_1[2], \ldots, t_n[2])$ but cannot both be included in $\mathrm{Eq}_{D'}$. □

Example 2. *We begin with a straightforward example from the Boolean domain. Define the binary relation R as $R = \{(0,1), (1,0), (1,1)\}$, and the unary hyperoperation h as $h(0) = \{0,1\}$, $h(1) = \{1\}$. Then h preserves R since $h(0,1) = \{(0,1), (1,1)\} \subseteq R$, $h(1,0) = \{(1,0), (1,1)\} \subseteq R$, and $h(1,1) = \{(1,1)\} \subseteq R$. However, h is not $\{0,1\}$-elementary since $|h(0)| = 2$, and Theorem 12 then implies that $\mathrm{Eq}_{\{0,1\}} \notin \langle \Gamma \rangle_{\neq}$. Using Theorem 8 and Theorem 10 we conclude that that $\mathcal{L}_{\langle \{R\} \rangle} = \{\langle \{R\} \rangle_{\neq}, \langle \{R, \mathrm{Eq}_{\{0,1\}}\} \rangle_{\neq}\}$ and that $|\mathcal{L}_{\langle \{R\} \rangle}| = 2$.*

For a more general example over $D = \{0,1,2\}$, let $\Gamma = \{R_1, R_2\}$ where $R_1 = \{0,1,2\}^2 \setminus \{(0,0)\}$ and $R_2 = \{0,1\}$, and consider the co-clone $\langle \Gamma \rangle$. We invite the reader to verify that $D' \in \langle \Gamma \rangle$ for a unary $D' \subset D$ if and only if $D' = \{0,1\}, D' = \{1,2\}$, or $D' = \{1\}$. Next, define the two hyperoperations h_1 and h_2 where: $h_1(0) = \{0,1\}$, $h_1(1) = \{1\}$, $h_1(2) = \{2\}$, and $h_2(0) = \{0\}$, $h_2(1) = \{1\}$, $h_2(2) = \{1,2\}$. It is then readily verified that $h_1, h_2 \in \mathrm{hPol}(\Gamma)$ but that h_1 is not $\{0\}$-elementary and that h_2 is not $\{2\}$-elementary. Hence, Theorem 12 implies that $\mathrm{Eq}_{\{0,1\}}, \mathrm{Eq}_{\{1,2\}}, \mathrm{Eq}_{\{0,1,2\}} \notin \langle \Gamma \rangle_{\neq}$, and we conclude that $\mathcal{L}_{|\langle \Gamma \rangle}$ consists of the five elements $\langle \Gamma \rangle_{\neq}$, $\langle \Gamma \cup \{\mathrm{Eq}_{\{0,1\}}\} \rangle_{\neq}$, $\langle \Gamma \cup \{\mathrm{Eq}_{\{1,2\}}\} \rangle_{\neq}$, $\langle \Gamma \cup \{\mathrm{Eq}_{\{0,1\}}, \mathrm{Eq}_{\{1,2\}}\} \rangle_{\neq}$, $\langle \Gamma \cup \{\mathrm{Eq}_{\{0,1,2\}}\} \rangle_{\neq}$.

4 Partial Restriction-Closed Hyperclones

In this section we consider the problem of describing restriction-closed hyperclones in the partial setting. We first observe that the set of all restriction-closed partial clones $\mathrm{hpPol}(\Gamma)$ is equal to the continuum even in the Boolean domain since it (trivially) includes all Boolean strong partial clones, whose cardinality is known to equal the continuum [1]. However, might it still be possible to describe the restriction closed partial hyperclones relative to the strong partial clones, similarly to how Theorem 8 and Theorem 10 revealed that the lattice of restriction-closed hyperclones is not significantly more complicated than the ordinary clone lattice? We thus begin by defining the partial analogue of Definition 6.

Definition 13. *For $\Gamma \subseteq \mathcal{R}_D$ over D we define the set $\mathcal{P}_{|\Gamma} = \{\langle \Delta \rangle_{\nexists \neq} \mid \Delta \subseteq \mathcal{R}_D, \langle \Delta \rangle_{\nexists} = \langle \Gamma \rangle_{\nexists}\}$.*

If we for the moment concentrate on the Boolean domain, then we would at least expect to find examples of $\mathcal{P}_{|\Gamma}$ consisting of two distinct elements $\langle \Gamma \rangle_{\neq \nexists}$ and $\langle \Gamma \cup \{\mathrm{Eq}_{\{0,1\}}\} \rangle_{\neq \nexists}$. However, all intervals that we have been able to identify either has only one element, or consists of at least three elements. A straightforward example of the former kind is the relation $R_{\leq} = \{(0,0),(0,1),(1,1)\}$, where $\mathrm{Eq}_{\{0,1\}}$ can be defined as $\mathrm{Eq}_{\{0,1\}}(x_1,x_2) \equiv R_{\leq}(x_1,x_2) \wedge R_{\leq}(x_2,x_1)$. For examples of the latter kind, let us consider two examples of such intervals for very simple languages Γ. Define $R_0 = \{0\}$, $R_{\neq} = \{(0,1),(1,0)\}$, and observe that $\mathrm{Pol}(\{R_{\neq}\})$ is the clone consisting of all Boolean, self-dual operations, while $\mathrm{Pol}(\{R_0\})$ consists of all Boolean operations f such that $f(0,\ldots,0) = 0$.

Theorem 14. $|\mathcal{P}_{|\{R_0\}}| \geq 3$ *and* $|\mathcal{P}_{|\{R_{\neq}\}}| \geq 3$.

Proof. We begin with R_0. Define the relation $R_0^{=} = \{(0,0,0),(0,1,1)\}$. Then we immediately have that $R_0(x) \equiv R_0^{=}(x,x,x)$ and $R_0^{=}(x_1,x_2,x_3) \equiv R_0(x_1) \wedge \mathrm{Eq}_{\{0,1\}}(x_2,x_3)$. We claim that the proper inclusions $\langle \{R_0\} \rangle_{\neq \nexists} \subset \langle \{R_0^{=}\} \rangle_{\neq \nexists} \subset \langle \{R_0, \mathrm{Eq}_{\{0,1\}}\} \rangle_{\neq \nexists}$ hold. Let $h_0(0,0) = \{0\}$, $h_0(0,1) = \{0,1\}$, and $h_0(x,y) = \emptyset$ otherwise. Then $h_0(0,0) = \{0\} \subseteq R_0$, and h_0 preserves R_0, but $h_0((0,0,0),(0,1,1)) = h_0(0,0) \times h_0(0,1) \times h_0(0,1) = \{0\} \times \{0,1\}^2 \nsubseteq R_0^{=}$. Similarly, let $h_1(0) = \emptyset$ and $h_1(1) = \{0,1\}$. Then $h_1(t) = \emptyset$ for $t \in R_0^{=}$, but $h_1((1,1)) = h_1(1) \times h_1(1) = \{0,1\}^2 \nsubseteq \mathrm{Eq}_{\{0,1\}}$, and h_1 does not preserve $\{R_0, \mathrm{Eq}_{\{0,1\}}\}$.

For $R_{\neq}$, define the relation

$$R(x_1,x_2,x_3,x_4) \equiv R_{\neq}(x_1,x_2) \wedge \mathrm{Eq}_{\{0,1\}}(x_3,x_4)$$

and observe that $R = \{(0,1,0,0),(0,1,1,1),(1,0,0,0),(1,0,1,1)\}$. We claim that $\langle \{R_{\neq}\} \rangle_{\neq \nexists} \subset \langle \{R\} \rangle_{\neq \nexists} \subset \langle \{R_{\neq}, \mathrm{Eq}_{\{0,1\}}\} \rangle_{\neq \nexists}$. First, it is straightforward to see that $R_{\neq} \in \langle \{R\} \rangle_{\neq \nexists}$ since

$$R_{\neq}(x_1,x_2) \equiv R(x_1,x_2,x_2,x_2).$$

Second, define the partial hyperoperation $h_2(0,0) = \{0,1\}$, $h_2(0,1) = \{0\}$, $h_2(1,0) = \{1\}$, and $h_2(1,1) = \emptyset$. Then $h_2((0,1,0,0),(1,0,0,0)) = h_2(0,1) \times h_2(1,0) \times h_2(0,0) \times h_2(0,0) = \{(0,1)\} \times \{0,1\}^2 \nsubseteq R$. Hence, h_2 does not preserve R. However, $h_2(t_1,t_2) \subseteq \{t_1,t_2\}$ for $t_1,t_2 \in R_{\neq}$ since $h_2(t_1,t_2) = \emptyset$ if $t_1 = t_2$, and $h_2(t_1,t_2) = \{t_1\}$ or $h_2(t_1,t_2) = \{t_2\}$ otherwise. For the inclusion $\langle \{R\} \rangle_{\neq \nexists} \subset \langle \{R_{\neq}, \mathrm{Eq}_{\{0,1\}}\} \rangle_{\neq \nexists}$ we simply define $h_3(0) = \{0,1\}$ and $h_3(1) = \emptyset$. Then $h_3((0,0)) = \{0,1\}^2 \nsubseteq \mathrm{Eq}_{\{0,1\}}$, but $h_3(t) = \emptyset \subseteq R$ for each $t \in R$. □

It is also not hard to come up with more complicated languages where $|\mathcal{P}_{|\Gamma}| \geq 3$ (but where it is even less obvious whether $|\mathcal{P}_{|\Gamma}| > 3$).

Define the relations $R_{1/3} = \{(0,0,1),(0,1,0),(1,0,0)\}$ and $R^{01=} = \{(0,1,0,0),(0,1,1,1)\}$. It can then be verified, with similar arguments as in the proof of Theorem 14, that $\langle R_{1/3}\rangle_{\neq\nexists} \subset \langle\{R_{1/3}, R^{01=}\rangle_{\neq\nexists} \subset \langle\{R_{1/3}, \mathrm{Eq}_{\{0,1\}}\}\rangle_{\neq\nexists}$.

5 Concluding Remarks

We have studied intervals of hyperclones, and managed to give a full description of all possible restriction-closed hyperclones induced by a given clone. Here, the main open question is to describe when $\mathrm{Eq}_{D_i} \in \langle \mathrm{Inv}(C) \cap \mathcal{IR}_D\rangle_{\neq}$, given a unary relation $D_i \in \mathrm{Inv}(C)$. Does there exist a general condition or is it something that has to be proven on an individual basis depending on the operations in C?

The question of describing the corresponding intervals of partial, restriction-closed hyperclones appears much harder, and it seems difficult to obtain a general and precise description of $|\mathcal{P}_{|\Gamma}|$. Hence, at this stage we have to settle by considering more fundamental questions. For example, does there exist Γ where $|\mathcal{P}_{|\Gamma}|= 2$, where $\mathcal{P}_{|\Gamma}$ is countably infinite, and where the cardinality of $\mathcal{P}_{|\Gamma}$ equals the continuum?

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