C-Maximal Strong Partial Clones and the Inclusion Structure of Boolean Weak Bases

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Strong partial clones are composition closed sets of partial operations containing all partial projections, characterizable as *partial polymorphisms* of sets of relations Γ ($\mathrm{pPol}(\Gamma)$). If C is a clone it is known that the set of all strong partial clones whose total component equals C, has a greatest element $\mathrm{pPol}(\Gamma_w)$, where Γ_w is called a *weak base*. Weak bases have seen applications in computer science due to their usefulness for proving complexity classifications for constraint satisfaction related problems. In this article we (1) completely describe the inclusion structure between $\mathrm{pPol}(\Gamma_w)$, $\mathrm{pPol}(\Delta_w)$ for all Boolean weak bases Γ_w and Δ_w and (2) in many such cases prove that the strong partial clones in question uniquely cover each other.

1 INTRODUCTION

A *clone* is a set of operations closed under composition which contains all projections. In the last decades clone theory has received quite some attention due to its relevance for classifying the complexity of computational problems such as *constraint satisfaction problems* (CSPs) [1]. This approach is based on the fact that a clone can be described as the set of *polymorphisms* of a set of relations, which, intuitively, may be viewed as generalisation of homomorphisms. Sets of polymorphisms then correspond to a closure operator on relations, closure under *primitive positive definitions*, which can be used to obtain reductions between CSPs. Not all computational problems are

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compatible with polymorphisms, in the sense that a clone corresponding to a constraint language unequivocally determines whether the problem is tractable, in P, or intractable (typically NP-hard or co-NP-hard). Some examples include the inverse satisfiability problem, enumerating models of CSP with polynomial delay, and the surjective CSP problem [8]. The complexities of several of these problems have been settled, but using non-algebraic proofs based on a large number of case analyses, only valid for the Boolean domain. Schnoor & Schnoor [18] argued that the complexity of such problems is better studied using *partial polymorphisms*, since these correspond to a more restricted closure operator on relations, closure under *quantifier-free primitive positive definitions*. These notions will be formally defined in Section 2 and at the moment we simply view the set of partial polymorphisms of a set of relations Γ, $\mathrm{pPol}(\Gamma)$, as polymorphisms that may be undefined for certain sequences of arguments. Unfortunately, the resulting closed classes of partial operations, *strong partial clones*, are largely unexplored even in the Boolean domain. To mitigate this Schnoor & Schnoor introduced the concept of a *weak base* [18] corresponding to a clone C, which is a relational description of the largest strong partial clone whose total component equals C. Furthermore, under some mild additional assumptions on the given clone C, they proved that a weak base can always be constructed in a systematic manner. Hence, if Γ_w is a weak base corresponding to a clone C then $\mathrm{pPol}(\Gamma_w)$ is the largest set of partial operations not containing a total operation outside of C. The practical motivation behind weak bases is that they offer a considerable simplification for proving hardness results, in the following sense. Assume that $X(\Gamma)$ is a computational problem parameterized by a set of relations Γ, and that we want to determine how Γ influences the complexity of $X(\Gamma)$. Then, instead of proving hardness results for $X(\Gamma)$ for every $\mathrm{pPol}(\Gamma)$ corresponding to a clone C, it is sufficient to show that $X(\Gamma_w)$ is intractable for a weak base Γ_w corresponding to C [18]. The reason is that Γ_w is the least expressive language corresponding to C with respect to quantifier-free primitive positive definitions, and $X(\Gamma_w)$ then intuitively represents the "easiest" problem corresponding to C. For arbitrary finite domains little is known concerning weak bases, but in the Boolean domain they are completely described [12], and have successfully been used to prove complexity dichotomies for several different computational problems [2, 3, 13, 18, 19].

In this article we study additional properties of Boolean weak bases, with a particular focus on their inclusion structure. More precisely, if we let $\mathcal{L}_{\mathcal{W}} = \{\mathrm{pPol}(\Gamma_w) \mid \Gamma_w \text{ is a Boolean weak base}\}$ then we are interested in determining the poset $(\mathcal{L}_{\mathcal{W}}, \subseteq)$. Such a classification can be of practical interest since it effectively reduces the number of distinct cases one needs to consider to prove a complexity dichotomy for a computational problem. Determining this inclusion structure is conceptually not difficult, but is in practice rather challenging due to the large number of cases that need to be considered. We propose a method where, given a weak base Γ_w corresponding to a clone C, one effectively needs to consider only the clones *covering* C, i.e., situated directly above in the clone lattice, rather than all clones containing C. Using this method, we in Section 3 completely describe the poset $\mathcal{L}_{\mathcal{W}}$.

In Section 4 we investigate additional properties of $\mathcal{L}_{\mathcal{W}}$, and are especially interested in determining which pairs of strong partial clones in $\mathcal{L}_{\mathcal{W}}$ (if any) that cover each other. While we do not obtain a complete dichotomy, we obtain several strong results, and are even able to provide examples of strong partial clones in $\mathcal{L}_W$ where one element uniquely covers the other. These covering proofs are based on describing the strong partial clones situated "close" to $\mathrm{pPol}(\Gamma_W)$, in the following sense. Let C be a clone and let Γ_W be a weak base corresponding to C. Say that $\mathrm{pPol}(\Gamma)$ is a C-*maximal* strong partial clone if $\mathrm{Pol}(\Gamma) = \mathsf{C}$ and $\mathrm{pPol}(\Gamma)$ is covered by $\mathrm{pPol}(\Gamma_W)$. Given two elements $\mathrm{pPol}(\Gamma_W)$ and $\mathrm{pPol}(\Delta_W)$ in $\mathcal{L}_{\mathcal{W}}$ where $\mathrm{pPol}(\Gamma_W) \subset \mathrm{pPol}(\Delta_W)$ we can then in many cases prove that $\mathrm{pPol}(\Delta_W)$ covers $\mathrm{pPol}(\Gamma_W)$ by comparing $\mathrm{pPol}(\Gamma_W)$ to the $\mathrm{Pol}(\Delta_W)$-maximal strong partial clones. For example, we prove that the strong partial clone $\mathrm{pPol}(\{\mathsf{WI}_2\})$, the set of all partial operations which cannot define a (non-projective) total operation, is covered by the strong partial clone $\mathrm{pPol}(\{\mathsf{WD}_1\})$ where $\mathsf{WD}_1 = \{(0,1,0,1),(1,0,0,1)\}$, but is not covered by any other strong partial clone. Here, it might also be interesting to observe that $\mathrm{pPol}(\{\mathsf{WD}_1\})$ is a so-called *submaximal* strong partial clone, i.e., it is covered by a maximal clone. Last, we wrap up the article by discussing continuations of this and other open questions in Section 5.

2 PRELIMINARIES

2.1 Partial Operations and Strong Partial Clones

A k*-ary partial operation* over a set D is a map $f\colon \mathrm{dom}(f) \to D$ where $\mathrm{dom}(f) \subseteq D^k$ ($k \geq 1$). We write PAR_D, respectively OP_D, for the set of all partial, respectively total, operations over the set D, and let $\mathsf{BF} = \mathrm{OP}_{\{0,1\}}$. If $f, g \in \mathrm{PAR}_D$, both of arity k, then g is a *suboperation* of f if $\mathrm{dom}(g) \subseteq \mathrm{dom}(f)$ and $g(\mathbf{x}) = f(\mathbf{x})$ for each $\mathbf{x} \in \mathrm{dom}(g)$. Partial operations compose together in a natural way, and if $f, g_1, \ldots, g_m \in \mathrm{PAR}_D$ are partial operations such that f has arity $m \geq 1$ and each g_i arity $n \geq 1$ then we write $f \circ g_1, \ldots, g_m$ for the n-ary partial operation

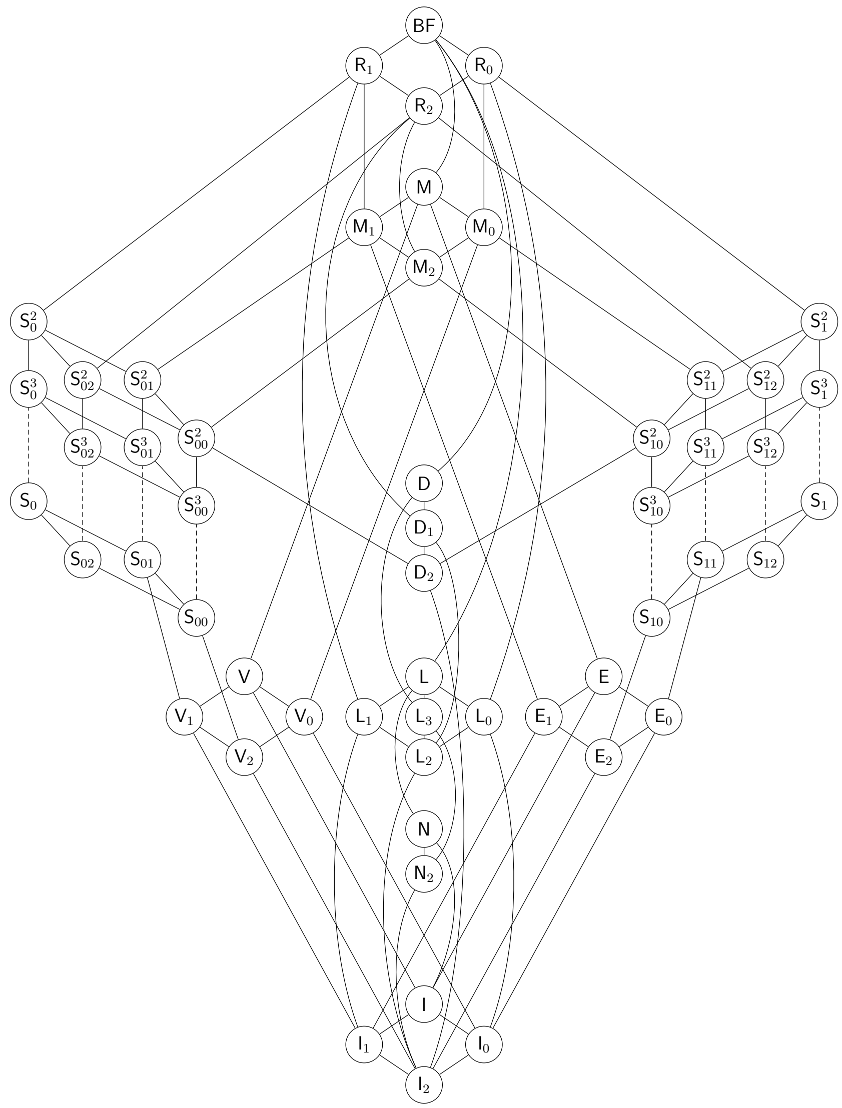

FIGURE 1: A visualization of Post's lattice of Boolean clones.

$f(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$ which is defined for $(x_1, \ldots, x_n) \in D^n$ if and only if

$$(x_1, \ldots, x_n) \in \bigcap_{1 \leq i \leq m} \mathrm{dom}(g_i)$$

and

$$(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n)) \in \mathrm{dom}(f).$$

Note that since a total operation can be viewed as a special case of a partial operation the above definition is valid also in the total setting. For $k \geq 1$ and $1 \leq i \leq k$ the *ith projection* π_i^k is defined as $\pi_i^k(x_1, \ldots, x_i, \ldots, x_k) = x_i$, and a suboperation of a projection is a a *partial projection*.

Definition 1. *A set* $\mathsf{C} \subseteq \mathrm{OP}_D$ *is a* clone *if* C *contains all projections over* D *and* C *is closed under composition, and a set* $\mathsf{P} \subseteq \mathrm{PAR}_D$ *is a* strong partial clone *if* P *contains all partial projections over* D *and* P *is closed under composition.*

If F is a set of (partial) operations then we write $[F]$ (respectively $[F]_s$) for the intersection of all (strong partial) clones containing F, and say that F is a *base*. If $F = \{f\}$ is singleton we write $[f]$ and $[f]_s$ instead of $[\{f\}]$ and $[\{f\}]_s$. If $\mathsf{C}_1 \subset \mathsf{C}_2$ are two (strong partial) clones, then C_2 is said to *cover* C_1 if there does not exist a (strong partial) clone C' such that $\mathsf{C}_1 \subset \mathsf{C}' \subset \mathsf{C}_2$, and we let $\mathrm{C}over(\mathsf{C})$ be the set of all (strong partial) clones covering C.

Our main interest in this article lies in studying Boolean (strong partial) clones. The cardinality of the lattice of Boolean strong partial clones is known to equal the continuum [9], while the lattice of Boolean clones, *Post's lattice*, is countable [16]. See Figure 1 for a visualization of Post's lattice and Table 1 for a comprehensive list of bases. Many of these bases are defined via Boolean expressions. For example, we write $\overline{x}$ for $f(0) = 1, f(1) = 0$, $x_1 \overline{\wedge} x_2$ for $f(0,0) = 1$, $f(0,1) = f(1,0) = f(1,1) = 0$, and $x_1 \leftrightarrow x_2$ for $f(0,0) = 1$, $f(0,1) = f(1,0) = 0$, $f(1,1) = 1$. In addition, we frequently write $x_1 \cdots x_n$ instead of $x_1 \wedge \ldots \wedge x_n$, and write 0 and 1 for the two constant Boolean operations. In addition, for each $n \geq 2$, we let $h_n(x_1, \ldots, x_{n+1}) = \bigvee_{i=1}^{n+1} x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1}$, and for each n-ary Boolean operation f, we let $\mathrm{dual}(f)(x_1, \ldots, x_n) = \overline{f(\overline{x_1}, \ldots, \overline{x_n})}$.

2.2 Partial Polymorphisms and Relations

Clones and strong partial clones can also be described through relations. First, let Rel_D be the set of all (finitary) relations over $D \subseteq \mathbb{N}$. Then, given a k-ary relation $R \in \mathrm{Rel}_D$ and an n-ary partial operation $f \in \mathrm{PAR}_D$ we say that f

TABLE 1: Bases of Boolean clones. The entries for S_0^n, S_{02}^n, S_{01}^n, S_{00}^n, S_1^n, S_{12}^n, S_{11}^n, S_{10}^n assume that $n \geq 2$.

C	Base of C
BF	$\{x \bar{\wedge} y\}$
R_0	$\{x \wedge y, x \oplus y\}$
R_1	$\{x \vee y, x \leftrightarrow y\}$
R_2	$\{x \vee y, x \wedge (y \leftrightarrow z)\}$
M	$\{x \vee y, x \wedge y, 0, 1\}$
M_0	$\{x \vee y, x \wedge y, 0\}$
M_1	$\{x \vee y, x \wedge y, 1\}$
M_2	$\{x \vee y, x \wedge y\}$
S_0^n	$\{x \rightarrow y, \mathrm{dual}(h_n)\}$
S_0	$\{x \rightarrow y\}$
S_{02}^n	$\{x \vee (y \wedge \neg z), \mathrm{dual}(h_n)\}$
S_{02}	$\{x \vee (y \wedge \neg z)\}$
S_{01}^n	$\{\mathrm{dual}(h_n), 1\}$
S_{01}	$\{x \vee (y \wedge z), 1\}$
S_{00}^n	$\{x \vee (y \wedge z), \mathrm{dual}(h_n)\}$
S_{00}	$\{x \vee (y \wedge z)\}$
S_1^n	$\{x \wedge \neg y, h_n\}$
S_1	$\{x \wedge \neg y\}$
S_{12}^n	$\{x \wedge (y \vee \neg z), h_n\}$
S_{12}	$\{x \wedge (y \vee \neg z)\}$
S_{11}^n	$\{h_n, 0\}$
S_{11}	$\{x \wedge (y \vee z), 0\}$
S_{10}^n	$\{x \wedge (y \vee z), h_n\}$
S_{10}	$\{x \wedge (y \vee z)\}$
D	$\{x\overline{y} \vee x\overline{z} \vee \overline{y}\overline{z}\}$
D_1	$\{xy \vee x\overline{z} \vee y\overline{z}\}$
D_2	$\{h_2\}$
L	$\{x \oplus y, 1\}$
L_0	$\{x \oplus y\}$
L_1	$\{x \leftrightarrow y\}$
L_2	$\{x \oplus y \oplus z\}$
L_3	$\{x \oplus y \oplus z \oplus 1\}$
V	$\{x \vee y, 0, 1\}$
V_0	$\{x \vee y, 0\}$
V_1	$\{x \vee y, 1\}$
V_2	$\{x \vee y\}$
E	$\{x \wedge y, 0, 1\}$
E_0	$\{x \wedge y, 0\}$
E_1	$\{x \wedge y, 1\}$
E_2	$\{x \wedge y\}$
N	$\{\overline{x}, 0, 1\}$
N_2	$\{\overline{x}\}$
I	$\{0, 1\}$
I_0	$\{0\}$
I_1	$\{1\}$
I_2	$\{\pi_1^1\}$

TABLE 2: Weak bases of Boolean co-clones. The entries for S_0^n, S_{02}^n, S_{01}^n, S_{00}^n, S_1^n, S_{12}^n, S_{11}^n, S_{10}^n assume that $n \geq 2$.

C	Weak base of IC
BF	$\{\mathrm{Eq}_{\{0,1\}}(x_1, x_2)\}$
R_0	$\{\mathrm{F}(c_0)\}$
R_1	$\{\mathrm{T}(c_1)\}$
R_2	$\{\mathrm{F}(c_0) \wedge \mathrm{T}(c_1)\}$
M	$\{(x_1 \rightarrow x_2)\}$
M_0	$\{(x_1 \rightarrow x_2) \wedge \mathrm{F}(c_0)\}$
M_1	$\{(x_1 \rightarrow x_2) \wedge \mathrm{T}(c_1)\}$
M_2	$\{(x_1 \rightarrow x_2) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)\}$
S_0^n	$\{\mathrm{OR}^n(x_1, \ldots, x_n) \wedge \mathrm{T}(c_1)\}$
S_0	$\{\mathrm{OR}^n(x_1, \ldots, x_n) \wedge \mathrm{T}(c_1) \mid n \geq 2\}$
S_{02}^n	$\{\mathrm{OR}^n(x_1, \ldots, x_n) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)\}$
S_{02}	$\{\mathrm{OR}^n(x_1, \ldots, x_n) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1) \mid n \geq 2\}$
S_{01}^n	$\{\mathrm{OR}^n(x_2, \ldots, x_{n+1}) \wedge (x_1 \rightarrow x_2 \cdots x_{n+1}) \wedge \mathrm{T}(c_1)\}$
S_{01}	$\{\mathrm{OR}^n(x_2, \ldots, x_{n+1}) \wedge (x_1 \rightarrow x_2 \cdots x_{n+1}) \wedge \mathrm{T}(c_1) \mid n \geq 2\}$
S_{00}^n	$\{\mathrm{OR}^n(x_2, \ldots, x_{n+1}) \wedge (x_1 \rightarrow x_2 \cdots x_{n+1}) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)\}$
S_{00}	$\{\mathrm{OR}^n(x_1, \ldots, x_n) \wedge (x \rightarrow x_1 \cdots x_n) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1) \mid n \geq 2\}$
S_1^n	$\{\mathrm{NAND}^n(x_1, \ldots, x_n) \wedge \mathrm{F}(c_0)\}$
S_1	$\{\mathrm{NAND}^n(x_1, \ldots, x_n) \wedge \mathrm{F}(c_0) \mid n \geq 2\}$
S_{12}^n	$\{\mathrm{NAND}^n(x_1, \ldots, x_n) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)\}$
S_{12}	$\{\mathrm{NAND}^n(x_1, \ldots, x_n) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1) \mid n \geq 2\}$
S_{11}^n	$\{\mathrm{NAND}^n(x_2, \ldots, x_{n+1}) \wedge (x_2 \rightarrow x_1) \wedge \ldots \wedge (x_{n+1} \rightarrow x_1) \wedge \mathrm{F}(c_0)\}$
S_{11}	$\{\mathrm{NAND}^n(x_2, \ldots, x_{n+1}) \wedge (x_2 \rightarrow x_1) \wedge \ldots \wedge (x_{n+1} \rightarrow x_1) \wedge \mathrm{F}(c_0) \mid n \geq 2\}$
S_{10}^n	$\{\mathrm{NAND}^n(x_2, \ldots, x_{n+1}) \wedge (x_2 \rightarrow x_1) \wedge \ldots \wedge (x_{n+1} \rightarrow x_1) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)\}$
S_{10}	$\{\mathrm{NAND}^n(x_2, \ldots, x_{n+1}) \wedge (x_2 \rightarrow x_1) \wedge \ldots \wedge (x_{n+1} \rightarrow x_1) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1) \mid n \geq 2\}$
D	$\{\mathrm{Neq}(x_1, x_2)\}$
D_1	$\{\mathrm{Neq}(x_1, x_2) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)\}$
D_2	$\{\mathrm{OR}^2(x_2, x_4) \wedge \mathrm{Neq}(x_2, x_3) \wedge \mathrm{Neq}(x_4, x_1) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)\}$
L	$\{\mathrm{EV}^4(x_1, x_2, x_3, x_4)\}$
L_0	$\{\mathrm{EV}^3(x_1, x_2, x_3) \wedge \mathrm{F}(c_0)\}$
L_1	$\{\mathrm{OD}^3(x_1, x_2, x_3) \wedge \mathrm{T}(c_1)\}$
L_2	$\{\mathrm{EV}^{3\neq}(x_1, \ldots, x_6) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)\}$
L_3	$\{\mathrm{EV}^{4\neq}(x_1, \ldots, x_8)\}$
V	$\{(\overline{x_4} \leftrightarrow \overline{x_2}\overline{x_3}) \wedge (\overline{x_2} \vee \overline{x_3} \rightarrow \overline{x_1})\}$
V_0	$\{(\overline{x_1} \leftrightarrow \overline{x_2}\overline{x_3}) \wedge \mathrm{F}(c_0)\}$
V_1	$\{(\overline{x_4} \leftrightarrow \overline{x_2}\overline{x_3}) \wedge (\overline{x_2} \vee \overline{x_3} \rightarrow \overline{x_1}) \wedge \mathrm{T}(c_1)\}$
V_2	$\{(\overline{x_1} \leftrightarrow \overline{x_2}\overline{x_3}) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)\}$
E	$\{(x_1 \leftrightarrow x_2 x_3) \wedge (x_2 \vee x_3 \rightarrow x_4)\}$
E_0	$\{(x_1 \leftrightarrow x_2 x_3) \wedge (x_2 \vee x_3 \rightarrow x_4) \wedge \mathrm{F}(c_0)\}$
E_1	$\{(x_1 \leftrightarrow x_2 x_3) \wedge \mathrm{T}(c_1)\}$
E_2	$\{(x_1 \leftrightarrow x_2 x_3) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)\}$
N	$\{\mathrm{EV}^4(x_1, x_2, x_3, x_4) \wedge x_1 x_4 \leftrightarrow x_2 x_3\}$
N_2	$\{\mathrm{EV}^{4\neq}(x_1, \ldots, x_8) \wedge x_1 x_4 \leftrightarrow x_2 x_3\}$
I	$\{(x_1 \leftrightarrow x_2 x_3) \wedge (\overline{x_4} \leftrightarrow \overline{x_2}\overline{x_3})\}$
I_0	$\{(\overline{x_1} \vee \overline{x_2}) \wedge (\overline{x_1}\overline{x_2} \leftrightarrow \overline{x_3}) \wedge \mathrm{F}(c_0)\}$
I_1	$\{(x_1 \vee x_2) \wedge (x_1 x_2 \leftrightarrow x_3) \wedge \mathrm{T}(c_1)\}$
I_2	$\{R_{1/3}^{\neq\neq\neq}(x_1, \ldots, x_6) \wedge \mathrm{F}(c_0) \wedge \mathrm{T}(c_1)\}$

TABLE 3: Relations.

Relation	Definition
F	$\{(0)\}$
T	$\{(1)\}$
Neq	$\{(0,1),(1,0)\}$
EV^n	$\{(x_1,\ldots,x_n) \in \{0,1\}^n \mid x_1+\ldots+x_n \text{ is even}\}$
$\mathrm{EV}^{n\neq}$	$\mathrm{EV}^n(x_1,\ldots,x_n) \wedge \mathrm{Neq}(x_1,x_{n+1}) \wedge \ldots \wedge \mathrm{Neq}(x_n,x_{2n})$
OD^n	$\{(x_1,\ldots,x_n) \in \{0,1\}^n \mid x_1+\ldots+x_n \text{ is odd}\}$
OR^n	$\{0,1\}^n \setminus \{(0,\ldots,0)\}$
NAND^n	$\{0,1\}^n \setminus \{(1,\ldots,1)\}$
$R_{1/3}^{\neq\neq\neq}$	$\{(0,0,1,1,1,0),(0,1,0,1,0,1),(1,0,0,0,1,1)\}$

preserves R, or that R is *invariant* under f, if for each sequence $t_1,\ldots,t_n \in R$ it holds that either

$$f(t_1,\ldots,t_n) := (f(t_1[1],\ldots,t_n[1]),\ldots,f(t_1[k],\ldots,t_n[k])) \in R$$

or that there exists i such that $(t_1[i],\ldots,t_n[i]) \notin \mathrm{dom}(f)$ (where $t_i[j]$ is the jth element of t_i).

If we then let $\mathrm{Pol}(\Gamma)$ (respectively $\mathrm{pPol}(\Gamma)$) be the set of all (partial) operations preserving each relation in Γ, it is easy to verify that $\mathrm{pPol}(\Gamma)$ forms a strong partial clone and that $\mathrm{Pol}(\Gamma)$ forms a clone. Dually, if $F \subseteq \mathrm{PAR}_D$, we let $\mathrm{Inv}(F) \subseteq \mathrm{Rel}_D$ (sometimes written $\mathsf{I}F$) be the set of all relations invariant under each (partial) operation in F. The operator $\mathrm{Inv}(\cdot)$ relate to $\mathrm{pPol}(\cdot)$ and $\mathrm{Pol}(\cdot)$ in the following sense.

Theorem 2 ([4, 5, 10, 17])**.** *Let Γ and Δ be two sets of relations over a finite set. Then (1) $\Gamma \subseteq \mathrm{Inv}(\mathrm{Pol}(\Delta))$ if and only if $\mathrm{Pol}(\Delta) \subseteq \mathrm{Pol}(\Gamma)$, and (2) $\Gamma \subseteq \mathrm{Inv}(\mathrm{pPol}(\Delta))$ if and only if $\mathrm{pPol}(\Delta) \subseteq \mathrm{pPol}(\Gamma)$.*

It is sometimes easier to work with $\mathrm{Inv}(F)$ directly instead of invoking its corresponding (strong partial) clone. Fortunately, these are well-behaved sets of relations, in the following sense: if F consists of total operations, then $\mathrm{Inv}(F)$ is closed under formation of first-order formulas consisting of existential quantification, conjunction, and equality constraints, *primitive positive definitions* (pp-definitions). To make this a bit more precise, first observe that the set of models of a first-order formula $\varphi(x_1,\ldots,x_n)$ can be viewed as a relation R, and we then write $R(x_1,\ldots,x_n) \equiv \varphi(x_1,\ldots,x_n)$ for $R = \{(f(x_1),\ldots,f(x_n)) \mid f \text{ is a model of } \varphi(x_1,\ldots,x_n)\}$. Then, if $\Gamma \subseteq \mathrm{Rel}_D$, a primitive positive definition of an n-ary $R \in \mathrm{Rel}_D$ over Γ is simply the condition that $R(x_1,\ldots,x_n) \equiv \exists y_1,\ldots,y_{n'} \colon R_1(\mathbf{x}_1) \wedge \ldots \wedge$

$R_m(\mathbf{x}_m)$ where each $R_i \in \Gamma \cup \{\mathrm{Eq}_D\}$ and each $\mathbf{x}_i$ is a tuple of variables over $x_1, \ldots, x_n, y_1, \ldots, y_{n'}$. Here, $\mathrm{Eq}_D = \{(x, x) \mid x \in D\}$ is the equality relation over D. Similarly, if $F \subseteq \mathrm{PAR}_D$ it is known that $\mathrm{Inv}(F)$ is closed under *quantifier-free primitive positive definitions* (qfpp-definitions) which are simply primitive positive definitions without existential quantification.

If we let $\langle\Gamma\rangle$ (respectively $\langle\Gamma\rangle_{\not\exists}$) be the smallest set of relations containing Γ and which is closed under pp-definitions (respectively, qfpp-definitions), then it is known that $\langle\Gamma\rangle = \mathrm{Inv}(\mathrm{Pol}(\Gamma))$ and that $\langle\Gamma\rangle_{\not\exists} = \mathrm{Inv}(\mathrm{pPol}(\Gamma))$. The sets $\langle\Gamma\rangle$ and $\langle\Gamma\rangle_{\not\exists}$ are furthermore known as *relational clones*, or *co-clones*, and *weak systems*, or *weak co-clones*. In both cases we refer to the set Γ as a *base* of $\langle\Gamma\rangle$ or $\langle\Gamma\rangle_{\not\exists}$. Our main usage of this correspondence in this article will be to show an inclusion of the form $\mathrm{pPol}(\Gamma) \subseteq \mathrm{pPol}(\Delta)$ by proving that each relation in Δ is qfpp-definable over Γ.

2.3 Intervals of Strong Partial Clones and Weak Bases

As remarked, the lattice of strong partial clones is of continuum cardinality even in the Boolean domain. The maximal elements have been determined [15][Section 20.4] and recently it was also proven that no minimal elements can exist [6], but in more general terms a complete understanding is still out of reach. A slightly more manageable strategy is to first fix a clone C and then describe the set of all strong partial clones corresponding to the clone C, motivating the following definition.

Definition 3. *Let* C *be a clone over a set* D*. We define the set* $\mathcal{L}_{D|\mathsf{C}} = \{\mathrm{pPol}(\Gamma) \mid \Gamma \subseteq \mathrm{Rel}_D, \mathrm{Pol}(\Gamma) = \mathsf{C}\}$.

Hence, $\mathcal{L}_{D|\mathsf{C}}$ is the set of all strong partial clones over D whose total component equals the given clone C. If the domain D is clear from the context, i.e., if the context is Boolean, then we for simplicity write $\mathcal{L}_{|\mathsf{C}}$ instead of $\mathcal{L}_{\{0,1\}|\mathsf{C}}$.

Say that a clone C is *finitely related* if there exists a finite $\Gamma \subseteq \mathrm{Rel}_D$ such that $\mathrm{Pol}(\Gamma) = \mathsf{C}$. Schnoor & Schnoor [18] proved that if C is finitely related then $\mathcal{L}_{D|\mathsf{C}}$ has a greatest element, namely the union of all members of $\mathcal{L}_{D|\mathsf{C}}$.

Theorem 4. *[18] Let* C *be a clone over a finite set* D*. If* C *is finitely related, then* $(\bigcup_{\mathsf{P} \in \mathcal{L}_{D|\mathsf{C}}}^{\infty} \mathsf{P}) \cap \mathrm{OP}_D = \mathsf{C}$.

The fact that a greatest element exists motivates the following definition.

Definition 5. *Let* C *be a clone over* D*. We say that* $\Gamma \subseteq \mathrm{Rel}_D$ *is a* weak base *of* $\mathrm{Inv}(\mathsf{C})$ *if* $\mathrm{pPol}(\Gamma) = (\bigcup_{\mathsf{P} \in \mathcal{L}_{D|\mathsf{C}}}^{\infty} \mathsf{P})$.

In relational terms Definition 5 then implies that $\langle\Gamma\rangle_{\nexists} \subseteq \langle\Delta\rangle_{\nexists}$ for each base Δ of $\mathrm{Inv}(\mathsf{C})$. Hence, a weak base is a base of $\mathrm{Inv}(\mathsf{C})$ minimally expressive with respect to qfpp-definitions. Boolean weak bases were fully described by Lagerkvist [12] and we refer the reader to Table 2 for a comprehensive list. Each entry consists of a Boolean clone C and a weak base of IC, typically represented via a logical formula. Variables are typically named $x_1, \ldots, x_n$ or x, y, z, with the exception of variables which are assigned constant values 0 and 1. These are instead denoted by c_0 and c_1, respectively, and we typically assume that c_0 occurs as the first argument and c_1 as the last. For example, the entry for the clone V in Table 2 consists of the logical formula $(\overline{x_4} \leftrightarrow \overline{x_2}\overline{x_3}) \wedge (\overline{x_2} \vee \overline{x_3} \rightarrow \overline{x_1})$ which defines the 4-ary relation $\{(0,0,0,0),(1,0,1,0),(1,1,0,0),(1,1,1,0),(1,1,1,1)\}$ which is a weak base of IV. Definitions of the additional relations used in Table 2 can be found in Table 3.

3 STRUCTURE OF BOOLEAN WEAK BASES

Given a Boolean weak base Γ_w, our goal is to describe every weak base Δ_w such that $\mathrm{pPol}(\Gamma_w) \subset \mathrm{pPol}(\Delta_w)$. To simplify the notation, given a Boolean clone C, we write WC for the weak base of $\mathrm{Inv}(\mathsf{C})$ from Table 2 and $\mathrm{P}\mathsf{C}$ for $\mathrm{pPol}(\mathsf{WC})$. Furthermore, let $\mathcal{L}_W = \{\mathrm{P}\mathsf{C} \mid \mathsf{C} \text{ is a Boolean clone}\}$. Before we turn to the technical details, we invite the reader to consult Figure 2, which is a visualization of the inclusions and non-inclusions in $\mathcal{L}_W$, later proven to be correct in Theorem 9.

Hence, our aim now is to describe the set $\mathcal{L}_W$ with respect to the partial order $\subseteq$. At a first glance this problem might appear to be straightforward due to Table 2 in combination with Post's lattice of Boolean clones [16]. In principle, what one needs to do is to, for every Boolean clone C and every Boolean clone C' such that $\mathsf{C} \subset \mathsf{C}'$, verify whether the inclusion $\mathrm{P}\mathsf{C} \subset \mathrm{P}\mathsf{C}'$ holds or not. This can be done by either showing that $\mathsf{WC}' \subset \langle\mathsf{WC}\rangle_{\nexists}$, implying that $\mathrm{P}\mathsf{C} \subset \mathrm{P}\mathsf{C}'$, or by finding a partial operation f preserving WC but not WC'. All inclusions of the former kind are in pratice rather straightforward to prove, and without further ado we present the majority of these definitions in Table 4 (the infinite chains in Post's lattice are handled later in Lemma 8). However, non-inclusions of the latter form are more troublesome, since we in the worst-case would need to compare $\mathrm{P}\mathsf{C}$ to $\mathrm{P}\mathsf{C}'$ for all pairs of Boolean clones where $\mathsf{C} \subset \mathsf{C}'$. Before we present our approach let us consider a concrete example.

Example 1. *Let us consider the clone* I_2*, which according to Table 2 only consists of projections. Then the strong partial clone* $\mathrm{P}\mathsf{I}_2$ *is largest strong partial clone in the interval* $\mathcal{L}_{|I_2}$*, and may therefore be viewed as the the largest Boolean strong partial clone which does not contain any total operations except for the projections. Since* $\mathsf{I}_2 \subseteq \mathsf{C}$ *for every Boolean clone, we therefore need to show that* $\mathrm{P}\mathsf{I}_2 \subseteq \mathrm{P}\mathsf{C}$*, or find a partial operation* f *preserving the weak base* WI_2 *but which does not preserve* WC*. Without any additional information available, it is then reasonable to start with the clones covering* I_2*, i.e., the minimal clones in Figure 1. Since it appears difficult to construct a qfpp-definition of* WC *using* WI_2 *for each such minimal clone* C *(according to the definitions in Table 2) we shift focus and instead try to prove that* $\mathrm{P}\mathsf{I}_2 \not\subseteq \mathrm{P}\mathsf{C}$*. For example,* $\mathrm{P}\mathsf{I}_2 \not\subseteq \mathrm{P}\mathsf{N}_2$ *since* $f(0,1) = f(1,0) = 0$ *preserves* WI_2 *but not* WN_2*, and it is indeed possible to show non-inclusion via similar partial operations for each minimal clone in Figure 2.*

However, this is far from sufficient, since these non-inclusions say nothing about whether $\mathrm{P}\mathsf{I}_2 \subseteq \mathrm{P}\mathsf{C}$ *for a non-minimal clone* C*. Thus, in the worst-case scenario we for* every *Boolean clone would need to provide a qfpp-definition or show non-inclusion by a partial operation. This is impractical already for* I_2*, and an insurmountable task if repeated for every Boolean clone.*

Thus, we need a method which avoids the tedious (and practically infeasible) case analysis between all possible pairs of Boolean clones. Let us illustrate how this can be achieved.

Example 2. *Let us return to* $\mathrm{P}\mathsf{I}_2$ *and* $\mathrm{P}\mathsf{N}_2$ *from Example 1, where we showed that* $\mathrm{P}\mathsf{I}_2 \not\subseteq \mathrm{P}\mathsf{N}_2$ *via the partial operation* $f(0,1) = f(1,0) = 0$ *which is included in* $\mathrm{P}\mathsf{I}_2$ *but not in* $\mathrm{P}\mathsf{N}_2$*. However, let us backtrack a bit, and for the moment assume that we are unaware of whether the inclusion* $\mathrm{P}\mathsf{I}_2 \subseteq \mathrm{P}\mathsf{N}_2$ *holds or not, and let us also remark that* $\mathsf{N}_2 = [\overline{x}]$ *(from Table 1). On the one hand, if* $\mathrm{P}\mathsf{I}_2 \subset \mathrm{P}\mathsf{N}_2$ *holds, then, trivially,* $[\mathrm{P}\mathsf{I}_2 \cup \{\overline{x}\}]_s \subseteq \mathrm{P}\mathsf{N}_2$*. On the other hand, if* $\mathrm{P}\mathsf{I}_2 \not\subseteq \mathrm{P}\mathsf{N}_2$ *then* $[\mathrm{P}\mathsf{I}_2 \cup \{\overline{x}\}]_s \not\subseteq \mathrm{P}\mathsf{N}_2$*, and it must be possible to construct a total operation* $f \in [\mathrm{P}\mathsf{I}_2 \cup \{\overline{x}\}]_s$ *not preserving* WN_2*. To see why we may assume that* f *is total, observe that if each total operation in* $[\mathrm{P}\mathsf{I}_2 \cup \{\overline{x}\}]_s$ *is included in* $\mathrm{P}\mathsf{N}_2$*, then* $[\mathrm{P}\mathsf{I}_2 \cup \{\overline{x}\}]_s \in \mathcal{L}_{|N_2}$*, implying that* $[\mathrm{P}\mathsf{I}_2 \cup \{\overline{x}\}]_s \subseteq \mathrm{P}\mathsf{N}_2$.

Hence, let us consider the expressive strength of $[\mathrm{P}\mathsf{I}_2 \cup \{\overline{x}\}]_s$*, and pick e.g. the binary and operation* $x \wedge y$*. How can we define this operation via composition using only partial operations from* $\mathrm{P}\mathsf{I}_2$ *and the total operation* $\overline{x}$*? This is clearly impossible if only unary partial operations from* $\mathrm{P}\mathsf{I}_2$ *are used, and we invite the reader to also verify that this is not possible if only binary* *partial operations from* PI_2 *are used. Hence, the intuition is that we should use at least one ternary partial operation* g *from* PI_2*, composed with* $\overline{x}$ *in such a way that the resulting operation is* $x \wedge y$*. For example, assume that we use the definition* $g(\overline{x}, x, y)$*. If we assume that the resulting operation defines* $x \wedge y$*, i.e.,* $x \wedge y = g(\overline{x}, x, y)$*, one can work backwards and conclude that* g *has to be defined as* $g(1,0,0) = g(1,0,1) = g(0,1,0) = 0$*,* $g(0,1,1) = 1$*. This partial operation does indeed preserve* WI_2*, and we conclude that* $x \wedge y \in [\mathrm{PI}_2 \cup \{\overline{x}\}]_s$*. The fact that* $x \wedge y \in [\mathrm{PI}_2 \cup \{\overline{x}\}]_s$ *does not preserve* WN_2 *may be seen as an alternative proof of* $\mathrm{PI}_2 \not\subseteq \mathrm{PN}_2$*, but is in fact a much stronger property since* $[\mathrm{PI}_2 \cup \{\overline{x}\}]_s = \mathsf{BF}$ *(since* $[\{x \wedge y, \overline{x}\}] = \mathsf{BF}$*). Thus,* $\mathrm{PI}_2 \not\subseteq \mathrm{PC}$ *for* every *Boolean clone* C *such that* $[\overline{x}] \subseteq \mathsf{C} \subset \mathsf{BF}$.

We may formalise the argument in Example 2 in the following theorem.

Theorem 6. *Let* $\mathsf{C}_1 \subset \mathsf{C}_3 \subseteq \mathsf{C}_2$ *be Boolean clones such that* $\mathsf{C}_3 \in Cover(\mathsf{C}_1)$*. If* $[\mathrm{PC}_1 \cup \mathsf{C}_3]_s \cap \mathsf{BF} \not\subseteq \mathsf{C}_2$*, then* $\mathrm{PC}_1 \not\subseteq \mathrm{PC}_2$*.*

Proof. If $\mathrm{PC}_1 \subset \mathrm{PC}_2$, then $\mathrm{PC}_2 \supseteq [\mathrm{PC}_1 \cup \mathsf{C}_3]_s$ since $\mathsf{C}_3 \subseteq \mathsf{C}_2$. But then WC_2 cannot be a weak base of $\mathrm{Inv}(\mathsf{C}_2)$ since $\mathrm{Pol}(\mathsf{WC}_2) \neq \mathsf{C}_2$ by the assumption that C_2 does not contain $[\mathrm{PC}_1 \cup \mathsf{C}_3]_s \cap \mathsf{BF}$. Hence, $\mathrm{PC}_1 \not\subseteq \mathrm{PC}_2$. □

As hinted, the advantage of Theorem 6 is therefore that we in practice only need to consider $Cover(\mathsf{C})$ instead of an arbitrary clone, in order to rule out possible inclusions in $\mathcal{L}_W$. Hence, for each Boolean clone C and $\mathsf{C}' \in Cover(\mathsf{C})$ we need to determine the strong partial clone $[\mathrm{PC} \cup \mathsf{C}']_s$. In other words we need to determine which total operations that are definable using partial polymorphisms of WC together with the new total operations from C'. To this aid we begin by defining the following.

Definition 7. *Let* $f, f_1, \ldots, f_m \in \mathrm{OP}_{\{0,1\}}$ *be operations of arity* n*. Define the* $(m+n)$*-ary partial operation* $g^f_{f_1,\ldots,f_m}$ *with domain*

$$domain(g^f_{f_1,\ldots,f_m}) = \{(f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}), \mathbf{x}) \mid \mathbf{x} \in \{0,1\}^n\},$$

such that

$$g^f_{f_1,\ldots,f_m}(f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}), \mathbf{x}) = f(\mathbf{x})$$

for each $\mathbf{x} \in \{0,1\}^n$.

The point of Definition 7 is therefore to construct a partial operation $g^f_{f_1,\ldots,f_m}$ using the given operations $f_1, \ldots, f_m$, such that f is included in $[\{f_1, \ldots, f_m, g^f_{f_1,\ldots,f_m}\}]_s$. In the case when some f_i does not depend on all its arguments, i.e., there exists $g_i \in [f_i]$ of arity less than n such that $f_i \in [g_i]$, we

TABLE 4: Qfpp-definition of WC_2 over WC_1.

C_2	C_1	$\mathsf{WC}_2 \in \langle\{\mathsf{WC}_1\}\rangle_{\not\exists}$
D_1	I_2	$\mathsf{WD}_1(c_0,x_1,x_2,c_1) \equiv \mathsf{WI}_2(c_0,c_0,x_1,x_2,c_1,x_2,c_1,c_1)$
R_0	I_0	$\mathsf{WR}_0(c_0) \equiv \mathsf{WI}_0(c_0,c_0,c_0,c_0)$
R_1	I_1	$\mathsf{WR}_1(c_1) \equiv \mathsf{WI}_1(c_1,c_1,c_1,c_1)$
M	I	$\mathsf{WM}(x_1,x_2) \equiv \mathsf{WI}(x_1,x_1,x_2,x_2)$
D	N_2	$\mathsf{WD}(x_1,x_2) \equiv \mathsf{WN}_2(x_1,x_2,x_1,x_2,x_1,x_2,x_1,x_2)$
S^2_{00}	V_2	$\mathsf{WS}^2_{00}(c_0,x_1,x_2,x_3,c_1) \equiv \mathsf{WV}_2(c_0,x_2,x_3,c_1,c_1)\wedge$
		$\mathsf{WV}_2(c_0,x_2,x_1,x_2,c_1) \wedge \mathsf{WV}_2(c_0,x_3,x_1,x_3,c_1)$
M_0	V_0	$\mathsf{WM}_0(c_0,x_1,x_2) \equiv \mathsf{WV}_0(c_0,x_1,x_2,x_2)$
S^2_{01}	V_1	$\mathsf{WS}^2_{01}(x_1,x_2,x_3,c_1) \equiv \mathsf{WV}_1(x_1,x_2,x_3,c_1,c_1)$
M	V	$\mathsf{WM}(x_1,x_2) \equiv \mathsf{WV}(x_2,x_2,x_1,x_1)$
S^2_{10}	E_2	$\mathsf{WS}^2_{10}(c_0,x_1,x_2,x_3,c_1) \equiv \mathsf{WE}_2(c_0,c_0,x_2,x_3,c_1)\wedge$
		$\mathsf{WE}_2(c_0,x_2,x_1,x_2,c_1) \wedge \mathsf{WE}_2(c_0,x_3,x_1,x_3,c_1)$
S^2_{11}	E_0	$\mathsf{WS}^2_{11}(c_0,x_1,x_2,x_3) \equiv \mathsf{WE}_0(c_0,c_0,x_3,x_2,x_1)$
M_1	E_1	$\mathsf{WM}_1(x_1,x_2,c_1) \equiv \mathsf{WE}_1(x_1,x_2,x_2,c_1)$
D_1	L_2	$\mathsf{WD}_1(c_0,x_1,x_2,c_1) \equiv \mathsf{WL}_2(c_0,c_0,x_1,x_1,x_2,x_2,c_1,c_1)$
D	L_3	$\mathsf{WD}(x_1,x_2) \equiv \mathsf{WL}_3(x_1,x_2,x_2,x_1,x_1,x_2,x_1,x_1,x_2)$
R_1	L_1	$\mathsf{WR}_1(c_1) \equiv \mathsf{WL}_1(c_1,c_1,c_1,c_1)$
R_0	L_0	$\mathsf{WR}_0(c_1) \equiv \mathsf{WL}_1(c_0,c_0,c_0,c_0)$
D_1	D_2	$\mathsf{WD}_1(c_0,x_1,x_2,c_1) \equiv \mathsf{WD}_2(c_0,c_1,x_1,c_0,x_2,c_1)$
R_2	D_1	$\mathsf{WR}_2(c_0,c_1) \equiv \mathsf{WD}_1(c_0,c_0,c_1,c_1)$
R_2	M_2	$\mathsf{WR}_2(c_0,c_1) \equiv \mathsf{WM}_2(c_0,c_0,c_0,c_1)$
R_1	M_1	$\mathsf{WR}_1(c_1) \equiv \mathsf{WM}_1(c_1,c_1,c_1)$
R_0	M_0	$\mathsf{WR}_0(c_0) \equiv \mathsf{WM}_0(c_0,c_0,c_0)$
M_2	S^2_{00}	$\mathsf{WM}_2(c_0,x_1,x_2,c_1) \equiv \mathsf{WS}^2_{00}(c_0,x_1,x_2,c_1,c_1)$
M_1	S^2_{01}	$\mathsf{WM}_1(x_1,x_2,c_1) \equiv \mathsf{WS}^2_{01}(x_1,x_2,c_1,c_1)$
R_2	S^2_{02}	$\mathsf{WR}_2(c_0,c_2) \equiv \mathsf{WS}^2_{02}(c_0,c_1,c_1,c_0)$
R_1	S^2_{0}	$\mathsf{WR}_1(c_1) \equiv \mathsf{WS}^2_{0}(c_1,c_1,c_1)$
M_2	S^2_{10}	$\mathsf{WM}_2(c_0,x_1,x_2,c_1) \equiv \mathsf{WS}^2_{10}(c_0,x_1,c_0,x_2,c_1)$
M_0	S^2_{11}	$\mathsf{WM}_0(c_0,x_1,x_2) \equiv \mathsf{WS}^2_{11}(c_0,x_1,c_0,x_2)$
R_2	S^2_{12}	$\mathsf{WR}_2(c_0,c_2) \equiv \mathsf{WS}^2_{12}(c_0,c_0,c_0,c_1)$
R_0	S^2_{1}	$\mathsf{WR}_2(c_0) \equiv \mathsf{WS}^2_{1}(c_0,c_0,c_0)$

will typically write $g^f_{f_1,\ldots,g_i,\ldots,f_m}$ instead of $g^f_{f_1,\ldots,f_i,\ldots,f_m}$ since the intended ordering of arguments will always be clear from the context. Let us illustrate how this construction can be used together with Theorem 6 by an example.

Example 3. *Consider the three clones* $\mathsf{BF}, \mathsf{I}_2, \mathsf{N}_2$*, and recall that* $\mathsf{BF} = [x \bar{\wedge} y]$, $\mathsf{I}_2 = [\pi^1_1]$*, and* $\mathsf{N}_2 = [\overline{x}]$*. In Example 2 we proved that* $\mathrm{P}\mathsf{I}_2 \cup \{\overline{x}\}$ *could define* $x \wedge y$*, but using Definition 7 it is also straightforward to show that* $x \bar{\wedge} y$ *can be defined, in a rather mechanical way. Thus, in order to apply Theorem 6, we show that* $x \bar{\wedge} y \in [\mathrm{P}\mathsf{I}_2 \cup \{\overline{x}\}]_s$*. Let* $f(x, y) = x \bar{\wedge} y$ *and* $f_1(x) = \overline{x}$*. Using Definition 7 we construct the ternary partial operation* $g^f_{f_1}$*, resulting in a partial operation with domain* $\{(f_1(x), x, y) \mid x, y \in \{0, 1\}\}$ *defined such that* $g^f_{f_1}(f_1(x), x, y) = x \bar{\wedge} y$ *for all* $x, y \in \{0, 1\}$*. In other words* $g^f_{f_1}(1, 0, 0) = 1$ *and* $g^f_{f_1}(1, 0, 1) = g^f_{f_1}(0, 1, 0) = g^f_{f_1}(0, 1, 1) = 0$*, and it is readily verified that* $g^f_{f_1}$ *preserves* WI_2*. Theorem 6 then implies that* $\mathrm{P}\mathsf{I}_2 \not\subseteq \mathrm{P}\mathsf{C}$ *for every clone* C *such that* $\mathsf{N}_2 \subseteq \mathsf{C}$ *and* $\mathsf{C} \neq \mathsf{BF}$*.*

The main technical difficulty is to choose the operations $f_1, \ldots, f_m$ in a suitable way such that the resulting partial operation $g^f_{f_1,\ldots,f_m}$ actually preserves WC. We have organised these definitions in Table 5, which should be interpreted as follows. First, each entry begins with three distinct clones $\mathsf{C}_1, \mathsf{C}_2, \mathsf{C}_3$ where $\mathsf{C}_3 \in Cover(\mathsf{C}_1)$ and $\mathrm{P}\mathsf{C}_1 \subset \mathrm{P}\mathsf{C}_2$. This is followed by one, or possibly two, operations f, f' such that $[\mathsf{C}_3 \cup \{f, f'\}] = \mathsf{C}_2$. The last element of the entry then consists of operations $f_1, \ldots, f_m, f'_1, \ldots, f'_m \in \mathsf{C}_3$ such that $g^f_{f_1,\ldots,f_m}$ and $g^{f'}_{f'_1,\ldots,f'_m}$ preserve $\mathsf{WC}_1{}^\star$. Hence, Theorem 6 implies that $\mathrm{P}\mathsf{C}_1 \not\subseteq \mathrm{P}\mathsf{C}'$ for any C' such that $\mathsf{C}_3 \subseteq \mathsf{C}' \subset \mathsf{C}_2$.

Example 4. *Consider the entry in Table 5 for* R_2*,* I_2*,* E_2*. Then* $f(x, y) = x \vee y$*,* $f'(x, y, z) = x \wedge (y \leftrightarrow z)$*, and* $\mathsf{E}_2 = [\wedge]$*. The provided definitions of* f_1*,* f'_1*, and* f'_2 *are* $f_1(x, y) = x \wedge y$*,* $f'_1(x, y, z) = x \wedge y$*, and* $f'_2(x, y, z) = x \wedge z$*, resulting in partial operations* $g^f_{f_1}$ *and* $g^{f'}_{f'_1,f'_2}$ *defined such that* $g^f_{f_1}(x \wedge y, x, y) = f(x, y) = x \vee y$ *and* $g^{f'}_{f'_1,f'_2}(f'_1(x, y, z), f'_2(x, y, z), x, y, z) = g^{f'}_{f'_1,f'_2}(x \wedge y, x \wedge z, x, y, z) = f'(x, y, z) = x \wedge (y \leftrightarrow z)$*. Hence,* $[\mathrm{P}\mathsf{I}_2 \cup \mathsf{E}_2]_s$ *contains* R_2*, and Theorem 6 then implies that* $\mathrm{P}\mathsf{I}_2 \not\subseteq \mathrm{P}\mathsf{C}$ *for every* $\mathsf{E}_2 \subseteq \mathsf{C} \subset \mathsf{R}_2$*.*

We now turn to the infinite chains in Post's lattice, i.e., clones C containing S_{00} but contained in S^2_0, or their dual clones S_{10} and S^2_1.

Lemma 8. *Let* $n \geq 2$*. Then* $\mathrm{P}\mathsf{S}^{n+1}_0 \subset \mathrm{P}\mathsf{S}^n_0$*,* $\mathrm{P}\mathsf{S}^{n+1}_{02} \subset \mathrm{P}\mathsf{S}^n_{02}$*,* $\mathrm{P}\mathsf{S}^{n+1}_{01} \subset \mathrm{P}\mathsf{S}^n_{01}$*,* $\mathrm{P}\mathsf{S}^{n+1}_{00} \subset \mathrm{P}\mathsf{S}^n_{00}$*, and* $\mathrm{P}\mathsf{S}^n_{00} \subset \mathrm{P}\mathsf{S}^n_{02}$*. Moreover,* $\mathrm{P}\mathsf{C} \not\subseteq \mathrm{P}\mathsf{C}'$ *for any other two clones* $\mathsf{C}, \mathsf{C}' \in \{\mathsf{S}^n_0, \mathsf{S}^n_{02}, \mathsf{S}^n_{01}, \mathsf{S}^n_{00} \mid n \geq 2\}$*.*

* The preservation condition has been formally verified by a computer program for all entries in the table.

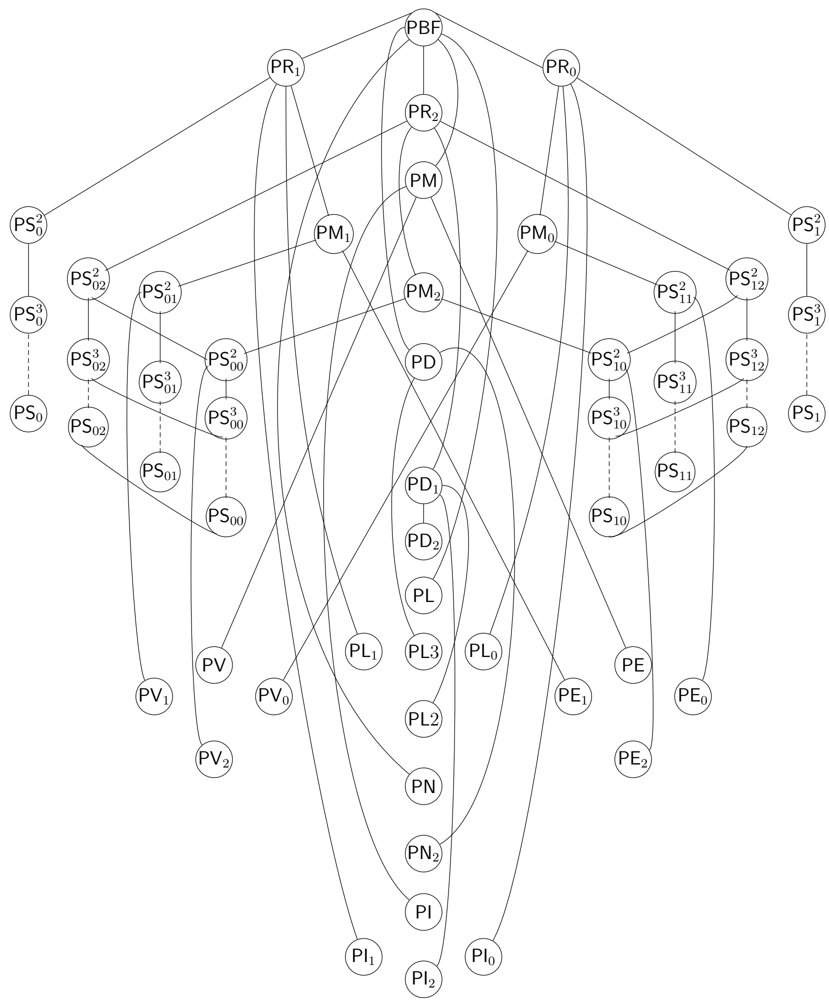

FIGURE 2: A visualization of the poset $(\mathcal{L}_{\mathcal{W}}, \subseteq)$. There exists a path consisting of upward edges connecting PC to PC' if and only if $\mathrm{PC} \subset \mathrm{PC}'$.

TABLE 5: Partial operations witnessing non-inclusions in Figure 2.

$\mathsf{C}_2, \mathsf{C}_1, \mathsf{C}_3$	f, f'	$(f_1, \dots, f_m), (f'_1, \dots, f'_m)$
$\mathsf{D}_1, \mathsf{I}_2, \mathsf{D}_2$	$xy \vee x\bar{z} \vee y\bar{z}$	$(h_2(x, y, z))$
$\mathsf{D}_1, \mathsf{I}_2, \mathsf{L}_2$	$xy \vee x\bar{z} \vee y\bar{z}$	$(x \oplus y \oplus z)$
$\mathsf{R}_2, \mathsf{I}_2, \mathsf{E}_2$	$x \vee y, x \wedge (y \leftrightarrow z)$	$(x \wedge y), (x \wedge y, x \wedge z)$
$\mathsf{R}_2, \mathsf{I}_2, \mathsf{V}_2$	$x \wedge y, x \wedge (y \leftrightarrow z)$	$(x \vee y), (x \vee y, x \vee z)$
$\mathsf{BF}, \mathsf{I}_2, \mathsf{I}_0$	$x \bar{\wedge} y$	(0)
$\mathsf{BF}, \mathsf{I}_2, \mathsf{I}_1$	$x \bar{\wedge} y$	(1)
$\mathsf{BF}, \mathsf{I}_2, \mathsf{N}_2$	$x \wedge y$	$(\overline{x})$
$\mathsf{R}_1, \mathsf{I}_1, \mathsf{V}_1$	$x \leftrightarrow y$	$(x \vee y)$
$\mathsf{R}_1, \mathsf{I}_1, \mathsf{L}_1$	$x \vee y$	$(x \leftrightarrow y)$
$\mathsf{R}_1, \mathsf{I}_1, \mathsf{E}_1$	$x \vee y, x \leftrightarrow y$	$(x \wedge y), (x \wedge y)$
$\mathsf{R}_1, \mathsf{I}_1, \mathsf{I}$	$x \vee y, x \leftrightarrow y$	$(0), (0)$
$\mathsf{R}_0, \mathsf{I}_0, \mathsf{E}_0$	$x \oplus y$	$(x \wedge y)$
$\mathsf{R}_0, \mathsf{I}_0, \mathsf{L}_0$	$x \wedge y$	$(x \oplus y)$
$\mathsf{R}_0, \mathsf{I}_0, \mathsf{V}_0$	$x \wedge y, x \oplus y$	$(x \vee y), (x \vee y)$
$\mathsf{R}_0, \mathsf{I}_0, \mathsf{I}$	$x \wedge y, x \oplus y$	$(1), (1)$
$\mathsf{M}, \mathsf{I}, \mathsf{V}$	$x \wedge y$	$(x \vee y)$
$\mathsf{M}, \mathsf{I}, \mathsf{E}$	$x \vee y$	$(x \wedge y)$
$\mathsf{M}, \mathsf{I}, \mathsf{N}$	$x \vee y, x \wedge y$	$(1), (1)$
$\mathsf{D}, \mathsf{N}_2, \mathsf{N}$	$xy \vee x\bar{z} \vee \bar{y}\bar{z}$	(1)
$\mathsf{D}, \mathsf{N}_2, \mathsf{L}_3$	$xy \vee x\bar{z} \vee \bar{y}\bar{z}$	$(x \oplus y \oplus z \oplus 1)$
$\mathsf{BF}, \mathsf{N}, \mathsf{L}$	$x \bar{\wedge} y$	$(x \oplus y)$
$\mathsf{BF}, \mathsf{V}_2, \mathsf{V}_1$	$x \bar{\wedge} y$	(1)
$\mathsf{BF}, \mathsf{V}_2, \mathsf{V}_0$	$x \bar{\wedge} y$	(0)
$\mathsf{S}_{00}^2, \mathsf{V}_2, \mathsf{S}_{00}$	$h_2(x, y, z)$	$(x \vee yz, y \vee xz, z \vee xy)$
$\mathsf{S}_{01}^2, \mathsf{V}_1, \mathsf{S}_{01}$	$h_2(x, y, z)$	$(x \vee yz, y \vee xz, z \vee xy)$
$\mathsf{S}_{01}^2, \mathsf{V}_1, \mathsf{V}$	$h_2(x, y, z)$	(0)
$\mathsf{M}_0, \mathsf{V}_0, \mathsf{V}$	$x \wedge y$	(1)
$\mathsf{S}_{10}^2, \mathsf{E}_2, \mathsf{E}_1$	$x \wedge (y \vee z), h_2(x, y, z)$	$(1), (1)$
$\mathsf{S}_{10}^2, \mathsf{E}_2, \mathsf{S}_{10}$	$h_2(x, y, z)$	$(xy \vee xz, yx \vee yz, zx \vee zy)$
$\mathsf{S}_{10}^2, \mathsf{E}_2, \mathsf{E}_0$	$x \wedge (y \vee z), h_2(x, y, z)$	$(0), (0)$
$\mathsf{M}_1, \mathsf{E}_1, \mathsf{E}$	$(x \wedge y)$	(0)
$\mathsf{S}_{11}^2, \mathsf{E}_0, \mathsf{S}_{11}$	$h_2(x, y, z)$	$(xy \vee xz, yx \vee yz, zx \vee zy)$
$\mathsf{S}_{11}^2, \mathsf{E}_0, \mathsf{E}$	$h_2(x, y, z)$	(1)
$\mathsf{BF}, \mathsf{L}_2, \mathsf{L}_0$	$x \bar{\wedge} y$	$(x \oplus y)$
$\mathsf{BF}, \mathsf{L}_2, \mathsf{L}_3$	$x \bar{\wedge} y$	$(y \oplus y \oplus x \oplus 1)$
$\mathsf{BF}, \mathsf{L}_2, \mathsf{L}_1$	$x \bar{\wedge} y$	$(x \leftrightarrow y)$
$\mathsf{D}, \mathsf{L}_3, \mathsf{L}$	$xy \vee x\bar{z} \vee \bar{y}\bar{z}$	(1)
$\mathsf{R}_1, \mathsf{L}_1, \mathsf{L}$	$x \vee y$	(1)
$\mathsf{R}_0, \mathsf{L}_0, \mathsf{L}$	$x \wedge y$	(1)
$\mathsf{D}_1, \mathsf{D}_2, \mathsf{S}_{00}^2$	$xy \vee x\bar{z} \vee y\bar{z}$	$(x \vee yz)$
$\mathsf{BF}, \mathsf{D}_1, \mathsf{D}$	$x \bar{\wedge} y$	$(xy \vee \bar{x}\bar{y})$
$\mathsf{BF}, \mathsf{R}_2, \mathsf{R}_1$	$x \bar{\wedge} y$	$(x \leftrightarrow y)$
$\mathsf{BF}, \mathsf{R}_2, \mathsf{R}_0$	$x \bar{\wedge} y$	$(x \oplus y)$

Proof. The inclusions can be proved via the qfpp-definitions:

$$\mathsf{WS}_0^n(x_1,\ldots,x_n,c_1) \equiv \mathsf{WS}_0^{n+1}(x_1,x_1,x_2,\ldots,x_n,c_1),$$

$$\mathsf{WS}_{02}^n(c_0,x_1,\ldots,x_n,c_1) \equiv \mathsf{WS}_{02}^{n+1}(c_0,x_1,x_1,x_2,\ldots,x_n,c_1),$$

$$\mathsf{WS}_{01}^n(x_1,\ldots,x_n,c_1) \equiv \mathsf{WS}_{01}^{n+1}(x_1,x_1,x_2,\ldots,x_n,c_1),$$

$$\mathsf{WS}_{00}^n(c_0,x_1,\ldots,x_n,c_1) \equiv \mathsf{WS}_{00}^{n+1}(c_0,x_1,x_1,x_2,\ldots,x_n,c_1),$$

and

$$\mathsf{WS}_{02}^n(c_0,x_1,\ldots,x_n,c_1) \equiv \mathsf{WS}_{00}^n(c_0,c_0,x_1,\ldots,x_n,c_1).$$

For a case $\mathrm{P}\mathsf{C} \not\subseteq \mathrm{P}\mathsf{C}'$ where inclusion does not hold we provide a partial operation f preserving WC' but not WC. Let f be the unary partial operation $f(1) = 0$. We claim that $f \in \mathrm{P}\mathsf{S}_{02}^k \setminus \mathrm{P}\mathsf{S}_0^n$, where $n \geq 2$. From Table 2 we see that $t[1] = 0$ for every $t \in \mathsf{WS}_{02}^k$, implying that $f(t)$ is always undefined and that f preserves WS_{02}^k. On the other hand, $1^n \in \mathsf{WS}_0^n$ but $0^n \notin \mathsf{WS}_0^n$, where $1^n = (1,\ldots,1)$ and $0^n = (0,\ldots,0)$ (both n-ary tuples) implying that $f(1^n) \notin \mathsf{WS}_0^n$. Using similar arguments it can be seen that $f \in \mathrm{P}\mathsf{S}_{00}^{\mathsf{k}} \setminus \mathrm{P}\mathsf{S}_0^n$ and $f \in \mathrm{P}\mathsf{S}_{00}^{\mathsf{k}} \setminus \mathrm{P}\mathsf{S}_{01}^{\mathsf{n}}$.

For the remaining case we define a binary partial operation f' such that $\mathrm{dom}(f') = \{(0,1),(1,0),(1,1)\}$ and $f'(0,1) = f'(1,0) = 0$, $f'(1,1) = 1$. From Table 2 we see that $\mathsf{WS}_{01}^k = \{(0,x_1,\ldots,x_n) | (x_1,\ldots,x_n) \in \{0,1\}^{k-1} \backslash 0^{k-1}\} \cup \{1^{k+1}\}$. This means that $f'(s,t)$ is defined for $s,t \in \mathsf{WS}_{01}^k$ only if there does not exist $i \in \{1,\ldots,k\}$ such that $s[i] = t[i] = 0$. Hence, if $f(s,t)$ is defined, then at least one of s and t is equal to 1^n. If $s = t = 1^k$, then $f'(s,t) = 1^k$, and if $s \neq t$, then from the definition of f' it must be the case that $f'(s,t) = s$ assuming $t = 1^k$ (the case when $s = 1^k$ is symmetric). This proves that $f' \in \mathrm{P}\mathsf{S}_{01}^k$. On the other hand, there exists $u,v \in \mathsf{WS}_0^n$ such that $u[i] \oplus v[i] = 1$ for $i \in \{1,\ldots,n\}$, and such that $u[n+1] = v[n+1] = 1$. This implies that $f'(u,v)$ is defined and returns a tuple w where $w[i] = 0$ for $i \in \{1,\ldots,n\}$, and where $w[n+1] = 1$. But then $w \notin \mathsf{WS}_0^n$. Hence, we conclude that f' preserves WS_{01}^k but not WS_0^n. □

Lemma 8 is also valid for $\mathrm{PS}_0, \mathrm{PS}_{02}, \mathrm{PS}_{01}, \mathrm{PS}_{00}$, and can be proved for the dual clones in Figure 1 using entirely analogous arguments. Finally, by combining the results in this section we may now prove the main result of the article (see Figure 2 for a visualization).

Theorem 9. *Let* C, C' *be two Boolean clones. Then* $\mathrm{P}\mathsf{C} \subset \mathrm{P}\mathsf{C}'$ *if and only if there exists a path consisting of upward edges connecting* $\mathrm{P}\mathsf{C}$ *to* $\mathrm{P}\mathsf{C}'$ *in Figure 2.*

Proof. All positive inclusions in Figure 2 follow from Table 4 and Lemma 8. Assume that $\mathrm{PC} \not\subseteq \mathrm{PC}'$ according to Figure 2 but that $\mathsf{C} \subset \mathsf{C}'$. If $\mathsf{S}_{00} \subseteq \mathsf{C} \subseteq \mathsf{S}_0^2$, or $\mathsf{S}_{10} \subseteq \mathsf{C} \subseteq \mathsf{S}_1^2$, then the non-inclusion follows from Lemma 8. Otherwise there exists an entry $\mathsf{C}_2, \mathsf{C}, \mathsf{C}_3$ in Table 5 such that $\mathsf{C}_3 \in Cover(\mathsf{C})$, $\mathsf{C}_3 \subseteq C'$, $f, f' \in \mathsf{C}_2$, $g^f_{f_1,\ldots,f_m}, g^{f'}_{f'_1,\ldots,f'_m} \in \mathrm{PC}$ and $f_1, f'_1, \ldots, f_m, f'_m \in \mathsf{C}_3$ such that $[\{g^f_{f_1,\ldots,f_m}, g^{f'}_{f'_1,\ldots,f'_m}, f_1, \ldots, f_m, f'_1, \ldots, f'_m\}]_s \subseteq [\mathsf{PC} \cup \mathsf{C}_3]_s \cap \mathsf{BF} \not\subseteq \mathsf{C}'$. Theorem 6 then gives the desired result that $\mathrm{PC} \not\subseteq \mathrm{PC}'$. □

4 COVERING AND C-MAXIMAL STRONG PARTIAL CLONES

Theorem 9 gives a complete classification of the inclusion structures of Boolean weak bases. However, several questions remain unanswered. For example, assume that there is an edge between PC_1 and PC_2 in Figure 2. This implies that $\mathrm{PC}_1 \subset \mathrm{PC}_2$, but does it also imply that PC_2 covers PC_1? We will see that this question can be related to the task of describing the strong partial clones in the "near vicinity" of PC, in the sense that we are interested in describing the strong partial clones covered by PC. Of particular interest are the maximal elements in the set $\mathcal{L}_{|\mathsf{C}}$, i.e., the strong partial clones in $\mathcal{L}_{|\mathsf{C}}$ covered by the greatest element PC. This naturally leads to the following definition.

Definition 10. *Let* C *be a Boolean clone. A strong partial clone* $\mathrm{pPol}(\Gamma) \in \mathcal{L}_{|\mathsf{C}}$ *is said to be* C-maximal *if* $\mathrm{pPol}(\Gamma)$ *is covered by* PC.

The definition can easily be generalised to larger domains, but to simplify the presentation we concentrate on the Boolean case. We can then relate the aforementioned covering question to C-maximal strong partial clones as follows.

Lemma 11. *Let* C_1 *and* C_2 *be two Boolean clones. If there exists a direct edge from* PC_1 *to* PC_2 *in Figure 2 and* PC_2 *does not cover* PC_1 *then either*

1) PC_1 *is included in a* C_2*-maximal strong partial clone, or*

2) there exists $\Delta_1, \Delta_2, \ldots$ *such that* $\mathrm{Pol}(\Delta_i) = \mathsf{C}_2$ *for each* $i \geq 1$, $\mathrm{PC}_1 \subset \mathrm{pPol}(\Delta_1) \subset \mathrm{pPol}(\Delta_2) \subset \ldots$, *and* $\bigcup_{i\geq 1} \mathrm{pPol}(\Delta_i) = \mathrm{PC}_2$.

Proof. If PC_2 does not cover PC_1 then there exists $\mathrm{pPol}(\Gamma)$ such that $\mathrm{PC}_1 \subset \mathrm{pPol}(\Gamma) \subset \mathrm{PC}_2$. Let $\mathrm{Pol}(\Gamma) = \mathsf{C}$. Clearly, $\mathsf{C} = \mathsf{C}_1$ cannot happen since PC_1 is the largest element in $\mathcal{L}_{|\mathsf{C}_1}$, and if $\mathsf{C}_1 \subset \mathsf{C} \subset \mathsf{C}_2$ then $\mathrm{PC}_1 \subset \mathrm{pPol}(\Gamma) \subseteq \mathrm{PC} \subset \mathrm{PC}_2$, contradicting the assumption that there exists a direct edge between PC_1 and PC_2 (if $\mathrm{PC}_1 \subset \mathrm{PC} \subset \mathrm{PC}_2$ then Figure 2 would not have an edge between PC_1 and PC_2). But then $\mathsf{C} = \mathsf{C}_2$, and if $\mathrm{pPol}(\Gamma)$ is not included in any maximal strong partial clone in $\mathcal{L}_{|\mathsf{C}_2}$ it is then clear that $\mathrm{pPol}(\Gamma)$ must be included in a chain of strong partial clones of the stated form. □

To prove covering between PC_1 and PC_2 we thus only have to rule out case 1) and case 2) in Lemma 11. In general, this is harder than one might expect since C-maximal elements are not well understood, but in practice this can be accomplished in surprisingly many cases due to the peculiar structure of the inclusions in Figure 2. In fact, we will be able to prove something significantly stronger for many strong partial clones PC. Say that PC_1 is *uniquely covered* by PC_2 if (1) PC_2 covers PC_1 and (2) there does not exist $\mathrm{pPol}(\Gamma) \neq \mathrm{PC}_2$ such that $\mathrm{pPol}(\Gamma)$ covers PC_1. The property of a strong partial clone uniquely covering a strong partial clone is incredibly strong, and in general we would expect such a property to hold only in rare cases. Before we investigate for which pairs of strong partial clones where unique covering holds we need the following lemma.

Lemma 12. *Let C_1 be a Boolean clone. If there exists a unique PC_2 such that there is an edge from PC_1 to PC_2 in Figure 2, and $\mathcal{L}_{|\mathsf{C}_2}$ is finite, then PC_1 is uniquely covered by PC_2 if $\mathrm{PC}_1 \not\subseteq \mathrm{pPol}(\Gamma)$ for each C_2-maximal strong partial clone $\mathrm{pPol}(\Gamma)$.*

Proof. Assume that $\mathrm{pPol}(\Gamma)$ covers PC_1, and let $\mathrm{Pol}(\Gamma) = \mathsf{C}$. If $\mathsf{C} \neq \mathsf{C}_2$ then $\mathrm{PC}_1 \subset \mathrm{pPol}(\Gamma) \subseteq \mathrm{PC}$, which contradicts the assumption that there exists a unique edge from PC_1 to PC_2. Hence, $\mathrm{Pol}(\Gamma) = \mathsf{C}$.

By assumption, $\mathcal{L}_{|\mathsf{C}_2}$ is finite, which rules out the existence of $\Delta_1, \Delta_2, \ldots$ such that $\mathsf{C}_2 = \mathrm{Pol}(\Delta_i)$ for each $i \geq 1$ and such that $\mathrm{pPol}(\Delta_1) \subset \mathrm{pPol}(\Delta_2) \subset \ldots$ and $\bigcup_{i \geq 1} \mathrm{pPol}(\Delta_i) = \mathrm{PC}_2$. But if $\mathrm{pPol}(\Gamma) \subset \mathrm{PC}_2$ then $\mathrm{pPol}(\Gamma)$ must be included in a C_2-maximal strong partial clone. Hence, if PC_1 is not included in any C_2-maximal strong partial clone then $\mathrm{pPol}(\Gamma) = \mathrm{PC}_2$, and PC_1 is uniquely covered by PC_2. □

A priori, Lemma 12 might not look very helpful since the assumption that $\mathcal{L}_{|C}$ is finite is very restrictive, but in practice this is true for a significant number of pairs PC_1, PC_2 in Figure 2, as the following theorem demonstrates.

Theorem 13. *The following statements are true.*

1) PD_1 *uniquely covers* PI_2, PD_2 *and* PL_2.

2) PD *uniquely covers* PL_3 *and* PN_2.

3) PM_2 *uniquely covers* PS^2_{00} *and* PS^2_{10}.

4) PM_1 *uniquely covers* PS^2_{01} *and* PE_1.

5) PM_0 *uniquely covers* PS^2_{11} *and* PV_0.

6) PM *uniquely covers* PI*,* PV*, and* PE.

7) PR_2 *uniquely covers* PS^2_{02}*,* PM_2*,* PD_1*, and* PS^2_{12}.

8) PR_1 *uniquely covers* PS^2_0*,* PI_1*,* PL_1*, and* PM_1.

9) PR_0 *uniquely covers* PS^2_1*,* PI_0*,* PL_0*, and* PM_0.

Proof. We begin by describing the C-maximal strong partial clones for each listed clone C of the first form, and remark that each interval $\mathcal{L}_{|\mathsf{C}}$ is then known to be finite. For D_1, Haddad & Simons fully classified $\mathcal{L}_{|\mathsf{D}_1}$ [11], and we have the D_1-maximal strong partial clones $\mathrm{pPol}(\{\mathsf{WD}_1, \mathrm{F}\})$, $\mathrm{pPol}(\{\mathsf{WD}_1, \mathrm{T}\})$, $\mathrm{pPol}(\{\mathrm{Neq} \times \mathrm{F}\})$, $\mathrm{pPol}(\{\mathrm{Neq} \times \mathrm{T}\})$, $\mathrm{pPol}(\{\mathrm{Neq}, \mathrm{F} \times \mathrm{T}\})$. Similarly, the M_2-maximal strong partial clones are $\mathrm{pPol}(\{\mathsf{WM}_2, \mathrm{F}\})$, $\mathrm{pPol}(\{\mathsf{WM}_2, \mathrm{T}\})$, $\mathrm{pPol}(\{\mathsf{WM}, \mathrm{F} \times \mathrm{T}\}\})$ [11]. The remaining intervals have a very simple structure, and it is known (see, e.g., Schölzel [20]) that the only M_1-maximal (respectively M_0-maximal) strong partial clone is $\mathrm{pPol}(\{\mathsf{WM}_1, \mathrm{T}\})$ (respectively $\mathrm{pPol}(\{\mathsf{WM}_0, \mathrm{F}\})$), that the R_2-maximal strong partial clones are $\mathrm{pPol}(\{\mathsf{WR}_2, \mathrm{F}\})$ and $\mathrm{pPol}(\{\mathsf{WR}_2, \mathrm{T}\})$, and that $|\mathcal{L}_{|R_0}| = |\mathcal{L}_{|R_1}| = |\mathcal{L}_{|\mathsf{M}}| = |\mathcal{L}_{|\mathsf{D}}| = 1$.

With these descriptions in mind the task is now straigthforward with the help of Lemma 12. Consider PD_1 and PI_2, and let the partial operations f_0, f_1, f_2, f_3, f_4 be defined as $f_0(0) = 1$, $f_1(1) = 0$, $f_2(0,0) = f_2(0,1) = f_2(1,0) = 0$, $f_3(0,1) = f_3(1,0) = f_3(1,1) = 1$, $f_4(0,1) = f_4(1,0) = 0$, and undefined otherwise. It is then readily verified that f_i for $0 \leq i \leq 4$ preserves WI_2 but that $f_0 \notin \mathrm{pPol}(\{\mathsf{WD}_1, \mathrm{F}\})$, $f_1 \notin \mathrm{pPol}(\{\mathsf{WD}_1, \mathrm{T}\})$, $f_2 \notin \mathrm{pPol}(\{\mathrm{Neq} \times \mathrm{F}\})$, $f_3 \notin \mathrm{pPol}(\{\mathrm{Neq} \times \mathrm{T}\})$, and that $f_4 \notin \mathrm{pPol}(\{\mathrm{Neq}, \mathrm{F} \times \mathrm{T}\})$. Hence, PI_2 is incomparable to each D_1-maximal strong partial clone, and it follows that PI_2 is uniquely covered by PD_1.

We consider one additional case in detail, namely PM_2 and PS^2_{00}. According to the aforementioned description of the PM_2-maximal strong partial clones, we need to show that PS^2_{00} is not included in $\mathrm{pPol}(\{\mathsf{WM}_2, \mathrm{F}\})$, $\mathrm{pPol}(\{\mathsf{WM}_2, \mathrm{T}\})$, and $\mathrm{pPol}(\{\mathsf{WM}, \mathrm{F} \times \mathrm{T}\}\})$. For the first two cases we may reuse the two partial operations f_0 and f_1 from the previous case. For the third case we define the ternary partial operation f_5 as $f_5(0,0,1) = 1$, $f_5(0,1,1) = 0$. Then f_5 does not preserve $\{\mathsf{WM}, \mathrm{F} \times \mathrm{T}\}$ since it does not preserve $\mathsf{WM} = \{(0,0), (0,1), (1,1)\}$, but is straightforward to verify that f_5

preserves WS^2_{00} since $f_5(t_1, t_2, t_3)$ is always undefined for any sequence of tuples $t_1, t_2, t_3 \in \mathsf{WS}^2_{00}$.

All other cases can be proven using similar arguments, and we omit the details. ☐

It might be interesting to observe that some cases of Theorem 13 were known to hold before [14], but in the context of determining the *submaximal* strong partial clones, i.e., strong partial clones covered by a maximal strong partial clone. One such example is PD_1 which is covered by the maximal strong partial clone $\mathrm{PR}_2 = \mathrm{pPol}(\mathrm{F} \times \mathrm{T})$.

5 CONCLUDING REMARKS

In this article we have fully described the inclusion structure of Boolean weak bases. In the process we also proved several strong covering results between PC_1 and PC_2 in Figure 2. An interesting continuation is to verify, or disprove, that an inclusion between PC and PC' in Figure 2 also implies that PC is covered by PC'. Here, one difficulty is that the remaining intervals $\mathcal{L}_{|C}$ are all equal to the continuum [7] and are in general not well understood, making it challenging to describe the C-maximal strong partial clones. Hence, is it possible to describe the C-maximal strong partial clones even if $\mathcal{L}_{|C}$ is not finite?

Another suitable topic is to study weak bases over arbitrary finite domains. In this setting we cannot hope for a complete classification akin to Figure 2, but even partial results could be of interest. For example, given a minimal clone C over a finite domain D, is it possible to describe a weak base of $\mathrm{Inv}(\mathsf{C})$?

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